An Inverse Problem Finite-Element-Method-Based Approach to the Hole Drilling Method in Photoelasticity

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ABSTRACT

The classical hole drilling method inverse problem in photoelasticity is generally defined in a biaxially loaded infinite plate in plane stress, and is dual in nature. One part of this duality relates to the problem of an infinite plate in which the circular hole is drilled first and then the biaxial loads are applied. The other is the residual stress problem, or hole drilling of an infinite plate after biaxial application of the loads. The objective in both of these experimental situations is the determination of the far-field or applied (residual) stresses from interpreting the isochromatic fringes orders around the hole. This paper addresses the problem of separation of stresses using the finite element method as the means to model the plate with a hole, rather than the Kirsch solution, and a least-squares optimization approach to resolve the applied stresses. A unique feature of this approach is that it is possible to use a finite plate width.

Introduction

Photoelasticity is a whole field optical technique whose application generally allows visualization of the stress field of a loaded two-dimensional transparent, birefringent model using a transmission polariscope. The stress field shows up as isochromatics or lines of maximum shear stress. In general, it is not straightforward to resolve the magnitudes and directions of the principal stresses from the isochromatics at every point of the material. One approach to obtaining the values and directions of the principal stresses is to create artificial surfaces in the material by drilling circular holes whose size is small relative to existing stress gradients. This is what is referred to as the hole method. The objective in this paper is to address the hole method problem using finite elements as the means to model the plate with a hole, and a least squares optimization approach to resolve for the applied stresses. Consideration of finite width plates is possible by this approach.

Figure 1 – Two equivalent approaches to general biaxial loading of an infinite plate with a hole
(a) Principal stresses at an angle from the horizontal
(b) Normal and shear stresses

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The Hole Method

The hole drilling method in photoelasticity is a classical inverse problem that is mostly known as a means of separation of stresses [1-5]. From an assessment of isochromatic fringes around the hole it is possible to determine the orientation and magnitudes of the principal stresses. This relies on the Kirsch solution, a forward problem, which postulates an uniaxially-loaded infinite plate with a hole whose solution yields the full-field stresses [6, 7]. This solution is the basis to generalize to the case of an infinite plate with a hole, shown in Figure 1(a), biaxially-loaded by principal normal stresses oriented at an angle to the horizontal direction, or, equivalently by the loading shown in Figure 1(b). Practically, this is the case where the circular hole is drilled first and then the loads are applied. The corresponding inverse problem is the resolution of the far-field stresses from the isochromatic fringes around the hole. The implications of successfully implementing separation of stresses include: (a) the stress field gradient is small with respect to the size of the hole, requiring that the stress gradients be fairly uniform over the areas of interest; (b) there is little loss of information close to the edge of the hole; and, (c) the photoelastic response of holes that sit side-by-side in the sample are independent. Typically this means that a finite plate needs to be at least 6.25 times as wide as the diameter of the hole for the Kirsch solution to be accurate to within 5% [8].

A second inverse problem that is less commonly addressed in photoelasticity is the residual stress problem [9-11]. The residual stress problem effectively corresponds to hole drilling after application of the loads. Determination of residual stresses by hole drilling is more typically resolved by using strain gages [12, 13]. The theoretical formulation of the residual stress problem is obtained by superposition. This amounts to the subtraction of the Kirsch-based solution of a general biaxial loading of an infinite plate with a hole, from the theoretical formulation for a general biaxial loading of an infinite plate.

The Finite Element Model

Figure 2 shows the discretized model used in implementing the finite element model. Symmetry considerations allow only a quarter of the plate to be used. The width of the plate is ten times the radius of the hole. The model uses eight-node isoparametric plane-stress elements, and includes 2450 elements and 7561 nodes. The left and lower boundaries of the model have been constrained to not move in the x- and y-directions, respectively. The right and upper boundaries are subjected to the applied normal stresses, \( \sigma_X \) and \( \sigma_Y \), respectively, and the shear stress, \( \tau_{XY} \). The element size ratio between elements on the outer boundaries and those near the hole is five. This element size ratio minimizes the error between the finite element and Kirsch solution results at the boundary of the hole for an applied equibiaxial stress state.

![Figure 2 – Schematic diagram of the finite element mesh and boundary conditions of the symmetric quarter-plate with a hole](image-url)
Optimization Approach

The optimization approach to be followed here parallels that used in the original work by Sanford and Dally [14-17], which has also been applied to the photoelastic hole method previously [11]. Briefly, assuming that we know the stress state at every point in the stress field of interest around the hole, we can write an objective function describing the relationship between calculated and experimental results, yielding

$$
[F_i(\sigma_X, \sigma_Y, \tau_{XY})]_{\text{experimental}} = [F_i(\sigma_X, \sigma_Y, \tau_{XY})]_{\text{numerical}} + e_i
$$

(1)

where, \([F_i(\sigma_X, \sigma_Y, \tau_{XY})]_{\text{experimental}}\) represents the experimental fringe order data at some point \((x_i, y_i)\) in the specimen; \([F_i(\sigma_X, \sigma_Y, \tau_{XY})]_{\text{numerical}}\) represents the finite element evaluation at the same point, with undetermined coefficients \((\sigma_X, \sigma_Y, \tau_{XY})\), the applied specimen stresses (residual stresses); and, \(e_i\) is the random error at each point. This equation implies that if the finite element program is able to exactly predict the experimental value at every point on the specimen the random error should be zero. In applying a least squares approach to this multi-valued expression the goal is to find a fit between the experimental data and the numerical finite element solution, so as to minimize the errors.

More specifically, the experimental values are represented by the fringe order data at each point in the specimen, i.e.,

$$
[(\tau_{max})_{i}]_{\text{experimental}} = \frac{(\sigma_y) - (\sigma_x)}{2} = \frac{n_i f_{\sigma}}{2t}
$$

(2)

where, \((\tau_{max})_i\), \((\sigma_1)_i\), and \((\sigma_2)_i\) are the maximum shear stress, the maximum principal stress and the minimum principal stress, respectively. \(n_i\) is the fringe order, \(f_{\sigma}\) is the material fringe value, and \(t\) is the plate thickness. From the finite element method, numerical values for the stress state at each corresponding point on the specimen are given by \((\sigma_x)_i\), \((\sigma_y)_i\), and \((\tau_{xy})_i\), from which values for maximum shear stress, \([((\tau_{max})_i)]_{\text{numerical}}\), are obtained. Note that lowercase subscripts are used for these local stress values. Previously, uppercase subscripts have been used to denote the far-field or applied stresses on the plate boundary. The maximum shear stress values obtained from finite elements are calculated from the expression,

$$
[((\tau_{max})_i)]_{\text{numerical}} = \sqrt{\left[\frac{(\sigma_y) - (\sigma_x)}{2}\right]^2 + [((\tau_{xy})_i)]^2}.
$$

(3)

The following functional results when Equation (1) is rearranged so as to be able to solve it in an overdeterministic sense,

$$
F_i(\sigma_X, \sigma_Y, \tau_{XY}) = \left\{\left[([\tau_{max}]_i)_{\text{numerical}}\right]^2 - \left[([\tau_{max}]_i)_{\text{experimental}}\right]^2\right\} = 0
$$

(4)

or,

$$
F_i(\sigma_X, \sigma_Y, \tau_{XY}) = \left\{\left[\frac{(\sigma_y) - (\sigma_x)}{2}\right]^2 + [((\tau_{xy})_i)]^2\right\} - \left\{\frac{n_i f_{\sigma}}{2t}\right\}^2 = 0.
$$

(5)

A Taylor’s series expansion of Equation (5) leads to,

$$
(F)_k = (F)_k + \left(\frac{\partial F}{\partial \sigma_x}\right)_k \Delta \sigma_x + \left(\frac{\partial F}{\partial \sigma_y}\right)_k \Delta \sigma_y + \left(\frac{\partial F}{\partial \tau_{xy}}\right)_k \Delta \tau_{xy}
$$

(6)

where \(k\) refers to the \(k\)th iteration step and \(\Delta \sigma_x\), \(\Delta \sigma_y\), and \(\Delta \tau_{xy}\) are corrections to the previous estimates of \(\sigma_x\), \(\sigma_y\), and \(\tau_{xy}\), respectively. The explicit relations for the correction terms are,

$$
\Delta \sigma_x = (\sigma_x)_{k+1} - (\sigma_x)_k; \quad \Delta \sigma_y = (\sigma_y)_{k+1} - (\sigma_y)_k; \quad \Delta \tau_{xy} = (\tau_{xy})_{k+1} - (\tau_{xy})_k
$$

(7)
from which the corrected estimates are obtained. If the experimental values and the numerically calculated values are close to each other this implies that \( F_i \) in Equation (6) equals to zero. Reordering Equation (6) and expressing it in matrix form we obtain, for \( m \) data points,

\[
\{F\} = [a]\{\Delta \sigma\} \quad (8)
\]

where,

\[
\{F\}_k = \begin{bmatrix} -F_1 \\ -F_2 \\ \vdots \\ -F_m \end{bmatrix}; \quad [a] = \begin{bmatrix} \frac{\partial F_1}{\partial \sigma_x} & \frac{\partial F_1}{\partial \tau_{xy}} & \cdots & \frac{\partial F_1}{\partial \tau_{xy}} \\ \frac{\partial F_2}{\partial \sigma_x} & \frac{\partial F_2}{\partial \tau_{xy}} & \cdots & \frac{\partial F_2}{\partial \tau_{xy}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial \sigma_x} & \frac{\partial F_m}{\partial \tau_{xy}} & \cdots & \frac{\partial F_m}{\partial \tau_{xy}} \end{bmatrix}; \quad \{\Delta \sigma\} = \begin{bmatrix} \Delta \sigma_x \\ \Delta \sigma_y \\ \Delta \tau_{xy} \end{bmatrix} \quad (9)
\]

Pre-multiplication of Equation (8) by \([a]^T\), and solving for the correction factors \(\{\Delta \sigma\}\), we obtain,

\[
\{\Delta \sigma\} = [c]^{-1}[a]^T\{F\} \quad (10)
\]

where,

\[
[c] = [a]^T[a] \quad (11)
\]

This iteration process is continued until acceptable convergence is attained, by defining a stopping criterion.

**Separation of stresses**

Figure 3 shows a flowchart outlining the procedures that were followed to verify that the inverse problem approach followed in this paper is a viable one. As the chart shows we have defined a specific problem, i.e., a separation of stresses problem and developed the forward FEM-based quarter-plate model mentioned above and schematically shown in Figure 3. The intent is to inversely solve for the far-field stresses from knowledge of the geometry and maximum shear stress field values. The material properties do not play a significant role in these calculations and therefore no mention is made of the actual values used in the FEM program. The chart in Figure 3 shows two distinct paths. One leads to the recursive inverse analysis described above, the other to a two-step procedure to verify that the FEM-based inverse problem solution works. The first step in this verification procedure requires that we define all the input parameters for use in the forward FEM-based model to generate an output data set. The output data set consists of the \((x, y)\) coordinates and the corresponding stress state given by \([\sigma_x, \sigma_y, \tau_{xy}]\), from which values for maximum shear stress, \([\tau_{\text{max}}]\), are obtained. A data file containing this output data is set aside for use as an input file to perform the recursive inverse analysis. Again, we would like to emphasize that the role of these initial simulations is to check out the overall approach to this FEM-Based inverse problem solution.

To start the needed inverse problem calculations we need to read in the output data that was previously generated, set the initialization parameters and perform the needed recursive analysis. Initially the intent is to verify convergence. If convergence is not achieved, the initialization parameters need to be modified until convergence is achieved. Next is the need to assure ourselves that the calculation convergence error is adequate. If this is successful, the results need to be verified. Since we know what initial data was used to generate the output results, it is easy to compare this assumed data to the results generated by this inverse problem procedure. The FEM-based inverse problem solution calculations were performed for three different cases of applied far-field stresses. Case 1 relates to application of only far-field stress \(\sigma_x\). Case 2 to application of far-field stresses \(\sigma_x\) and \(\sigma_y\). And, Case 3 to applied far-field stresses \(\sigma_x\), \(\sigma_y\) and \(\tau_{xy}\). Table 1 shows the values of the far-field stresses and the corresponding initial guesses that were used to run the corresponding FEM-based inverse problem.
solution calculations. Sometimes this required making several attempts at initial guesses so as to verify convergence of the inverse analysis. Once this was achieved it was possible, in all cases, to achieve convergence to the appropriate values, with an error of the order of the assumed convergence error, which was arbitrarily set at $1 \times 10^{-8}$ MPa.

![Flowchart](image)

Figure 3 – Flowchart outlining the finite-element-method-based verification approach to moiré hole drilling

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**Summary and Conclusions**

In summary, several objectives have been accomplished in this paper: (1) re-examination of the duality of the hole method inverse problem in photoelasticity, i.e., consideration of the hole method in photoelasticity as a means of separation of stresses and for residual stress determination; (2) re-examination of the hole method inverse problem in photoelasticity using an FEM-based approach both with simulated results to test out the proposed scheme; (3) this approach may be extended to finite width plates with holes, or for that matter to many other problems of interest to practitioners by the general principles outlined in this paper. It can be anticipated that the presently outlined procedures might have wider applicability when used with other experimental techniques such as moiré, holographic interferometry, Digital Image Correlation, ESPI, etc.
References

13. TN-503-1: Measurements of residual stresses by the hole drilling strain-gage method. 1985, Measurements Group: Raleigh, NC.