ERROR LOCALIZATION FOR UPDATING FINITE ELEMENT MODELS USING FREQUENCY-RESPONSE-FUNCTIONS

by

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ABSTRACT

In the present article, the authors develop a new formulation for the problem of updating Finite Element models based upon results obtained from experimental tests. The process of updating is iterative and divided in two stages, (i) localization of the zones where the highest errors between the theoretical model and the experimental results occur and (ii) correction of the theoretical model in those zones. This paper focuses on the first stage, the error localization, the technique developed being based upon the definition of an error measure on the constitutive relation, which takes into account the forced response of a mechanical system and consequently enabling the incorporation of the frequency-response-functions obtained from experiments. In some previous works, the use of an error measure on the constitutive relation has been shown as highly efficient in the whole process of Finite Element updating, when free vibration was considered and only the experimental resonant frequencies accounted for. The work presented here is, somehow, an extension of those others. At present, the models only accommodate undamped or very lightly damped systems, future extensions being sought. A numerical example is given, where various situations are simulated, in order to validate the method.

1 INTRODUCTION AND OBJECTIVES

Due to the increasing importance of the updating of finite element models based upon results from experimental tests, there exists nowadays a large number of techniques available for such a purpose. Here, we shall suppose that the structure under consideration may be a large one, discarding the possibility of a global resolution of the problem, which would consider the simultaneous optimization of all the design parameters. The most reasonable strategy consists of localizing, in a first stage, the erroneous regions of the theoretical model and, in a second stage, correcting the design parameters for those zones, taking into account the physics of the structure.

In the present work a new approach is addressed, based upon the establishment of a distance between theory and tests through the definition of an error measure on the constitutive relation, introduced by Ladevèze (Refs. [1] to [3]), and followed by other authors (Refs. [4] to[6]). Those authors formulate the updating problem in terms of free vibrations, thus taking from the experimental tests the information concerning the natural frequencies and the corresponding mode-shapes. As the natural frequencies are obtained, generally, with good accuracy, these are considered as being exact. The mode-shapes, due to their local character, are usually obtained with less accuracy and so, into the formulation of the problem, they are taken with a certain weighting factor. For the localization of the zones wrongly modeled, a functional is built, where the error on the constitutive relation is minimized, taking also into account the "distance" between the theoretical and experimental mode-shapes and verifying the dynamic equilibrium equation for free vibration.

The objective of this article is to develop a new approach to the problem of localizing the wrongly modeled zones of a structure, following the same philosophy of minimizing the error on the constitutive law, but exploring further the experimental results, taking into consideration not only the natural frequencies and mode-shapes, but all the information in terms of frequency-response-functions (FRFs). Therefore, we shall be dealing with forced vibrations instead of free vibrations, which implies the overall reformulation of the problem.

The effort of developing a new formulation for the updating of Finite Element models advises a cautious start. Consequently, in this first paper, we shall focus our study on the localization stage of the problem. Furthermore, damping will not be considered, which makes the present approach only suitable for lightly damped structures and the errors will be supposed to concentrate only in the stiffness matrix.

In the next section, a very brief review of some of the most important Finite Element updating techniques is given. In section 3, the bases of the existing techniques based on the
notion of error in the constitutive relation are briefly reminded, in order to adequately frame the present development. Section 4 is dedicated to the description of the new approach and constitutes the main part of this article. Finally, in section 5, we shall present some numerical examples and, in section 6, the main conclusions.

2 BRIEF REVIEW OF PREVIOUS WOKS


2.1 Comparison methods

Comparison methods are simple indicators of the existing scatter between analytical and experimental modes shapes and not really updating techniques. They indicate which are the "good" and the "bad" modes in the model, but say nothing about where or how to correct the model. In this category there are the "Modal Scale Factor" [8], the "Cross Orthogonality Method" [9], the "Pseudo Cross Orthogonality Check" [10], the "Mixed Orthogonality Check" [8], [9], the "Correlation Coefficients" [11] and the very popular "Modal Assurance Criterion" [12].

2.2 Localization methods

Within the category of Localization methods, one can find the method proposed by Brown [13] and another one by He [14], but surely the most interesting and used one is the "Coordinate Modal Assurance Criterion", due to Lieven [15], [7].

2.3 Correlation methods

Correlation methods are, no doubt, the most significant, complete and important methods, generally comprehending the two necessary phases for a good model updating: the localization and the correction. Here, we extend the classification of Lieven, subdividing this category into five subcategories, (i) Matrix Methods, (ii) Sensitivity Methods, (iii) Frequency Domain Methods, (iv) Dynamic Reactions Methods and (v) Constitutive Relation Error Based Methods.

2.3.1 Matrix methods

These kind of methods manipulate directly the errors between theoretical and experimental mass and stiffness matrices. They supply the global modifications to the theoretical matrices, in order to update them. In Refs. [16] to [20] several approaches are presented, but probably the most popular and practical technique is the one known as "Error Matrix Method", first presented by Sidhu and Ewins (Ref. [21]), certainly due to the fact that the information concerning the errors is dominated by the low frequency modeshapes, relatively easy to obtain.

2.3.2 Sensitivity methods

This type of methods is characterized by the use of a Jacobian, which is applied to the correction parameters, where a first order Taylor series expansion is considered. Usually, it is necessary to undertake a previous matching between the analytical and experimental modes shapes. One can cite the methods developed by Zhong and Lallmenn (Refs. [22] to [24]). Other works are also to be mentioned, see Refs. [25] to [32]. In Ref. [33], Ibrahim gives a different approach, where only two experimental modes shapes are used, in a direct, non-iterative way.

2.3.3 Frequency Domain methods

These methods use directly the Frequency Response Functions instead of the modal parameters, to build either an error matrix in the frequency domain ([34], [35]), where the correction parameters are found through the solution of a linear system of equations, or an error functional between the experimental and analytical FRFs ([36], [37]), where both stages of localization and correction are considered, this last one solved separately for each degree-of-freedom.

2.3.4 Dynamic Reactions methods

This type of methods (Refs. [38] to [40]) explore the residues in the variational expression of the equilibrium equations and also the experimental modes shapes, in the resolution of a series of linear static problems. The eigenvalues and eigenvectors obtained from the experiments are considered as being exact, the difference between analytical and experimental results being expressed by non-zero reaction forces over the degrees-of-freedom of the structure. The calculation of the correcting parameters represents, at each iteration, a quadratic optimization problem without constraints.

2.3.5 Constitutive Relation Error Based methods

These methods, exploring the error in the constitutive relation (Refs. [2], [3], [4], [6]), are the basis for the development presented in this article and, as aforementioned, will be explained in section 3, while in section 4 the new approach will be proposed, using the measured FRFs.

3 THE UPDATING PROBLEM BASED ON THE NOTION OF ERROR IN THE CONSTITUTIVE RELATION

The model proposed by the above cited authors considers the free vibration case, without damping and under the hypothesis of small perturbations in elasticity. The natural frequencies and the corresponding modes shapes are supposed to have been extracted from the experimental tests.

In order to correctly formulate the problem, it is convenient - prior to the presentation of the updating problem applied to the finite element discretized structure - to make a formal development in terms of continuum mechanics.

3.1 Continuum mechanics formulation

For the model of the real structure, the reference solution is constituted by a displacement field verifying the kinematic
constraints and a stress field verifying the equilibrium equation. The data, for the free vibration case, are:
- The theoretical modeling;
- The experimental eigenvalues \( \lambda_i \), \( i \in [1, q] \);
- The experimental eigenvectors \( \psi_i \), from which only a subset \( \mathcal{P}_\psi \) is known, \( \mathcal{P} \) being the corresponding projection operator.

Let us consider a bounded domain \( \Omega \), with the boundary \( \partial \Omega \) corresponding to the structure, and two subsets, \( \partial_1 \Omega \) where the displacement field is imposed and \( \partial_2 \Omega \), the complementary part, where the normal stress vector is imposed (Fig. 1).

![Fig. 1](image_url)

In the theoretical case, for each eigenvalue \( \lambda \), the problem of finding the corresponding eigenvector is stated in the following way:

Find the couple \((v, \sigma)\), where \( v \) is a displacement field and \( \sigma \) a stress field, such that

\[
v \in \mathbf{V} \quad \mathbf{V} = \{ v : v|_{\partial_\Omega} = 0 \text{ and } v \text{ regular} \} \tag{1}
\]

\[
\forall v^* \in \mathbf{V} \quad \int_\Omega \text{Tr}(\sigma \varepsilon(v^*)) \, d\Omega = \lambda \int_\Omega \rho v^* \, d\Omega \tag{2}
\]

and \( v \) and \( \sigma \) verifying the constitutive relation:

\[
\sigma = \mathcal{H} \varepsilon(v) \tag{3}
\]

where \( \rho \) is the density, \( \varepsilon \) the strain operator and \( \mathcal{H} \) the Hooke operator.

In practice, we shall work with the experimental eigenvalues \( \lambda_i \). As a consequence, there will be an error in the constitutive relation (3), which can be expressed as:

\[
J(v, \sigma) = \left\| \sigma - \mathcal{H} \varepsilon(v) \right\|^2 = \int_\Omega \text{Tr}\left( (\sigma - \mathcal{H} \varepsilon(v)) \mathcal{H}^{-1} (\sigma - \mathcal{H} \varepsilon(v)) \right) \, d\Omega \tag{4}
\]

or, in terms of a displacement field \( u \in \mathbf{V} \) associated to the stress \( \sigma \),

\[
J(v, \sigma) = \|u - v\|^2 = \int_\Omega \text{Tr}[\mathcal{H} \varepsilon(u - v) \varepsilon(u - v)] \, d\Omega \tag{5}
\]

Therefore, the problem is:

Find \((u', v')\) minimizing the error

\[
\|u - v\|^2 = \int_\Omega \text{Tr}[\mathcal{H} \varepsilon(u - v) \varepsilon(u - v)] \, d\Omega
\]

submitted to the constraints

\[
u \in \mathbf{V}, \quad v \in \mathbf{V} \quad \mathbf{V} = \{ v : v|_{\partial_\Omega} = 0 \text{ and } v \text{ regular} \}
\]

\[
\forall v^* \in \mathbf{V} \quad \int_\Omega \text{Tr}(\mathcal{H} \varepsilon(u) \varepsilon(v^*)) \, d\Omega = \lambda \int_\Omega \rho v^* \, d\Omega
\]

As one wishes to avoid the solution \((0,0)\) and the experimental modeshapes corresponding to \( \lambda \) are available (which we have denoted by \( \mathcal{P}_\psi \)), one may associate \((u', v')\) to the theoretical model and to the experimental mode \((\lambda, \mathcal{P}_\psi)\), solving the following problem:

Find \((u', v')\) minimizing the error

\[
e^2(u, v) = \|u - v\|^2 + \frac{r}{1 - r} \left\| \mathcal{P} u - \mathcal{P} v \right\|^2 \tag{7}
\]

submitted to the same constraints as in (6).

where \(\| \cdot \|\) represents a norm in the subspace where \( y \) is known and \( r \) is a scalar representing the confidence on the experimental results, \( r < 1 \) when they are not very reliable, \( r \) close to 1 in the opposite case. A common value is 0.5. When experimental results are particularly good, it is usual to take \( r = 0.7 \).

3.1.1 Error indicators

For a structure constituted by several sub-structures \( (s) \), the accuracy of the model at each sub-structure may be expressed as:

\[
e_{s}^2 = \frac{1}{2} \left( \frac{\|u - v\|^2}{\|v\|^2} \right) \tag{8}
\]

As the displacement levels may be very different, depending on the modeshape, it is advisable to consider the summation of the \( e_{s}^2 \) contributions for the \( q \) measured modeshapes. Hence, for each sub-structure \( s \),

\[
\Gamma_{s} = \sum_{k=1}^{q} \frac{1}{2} \left( \frac{\|u - v\|^2}{\|v\|^2} \right)_{k}
\]

Other indicators have been proposed [4], being able to quantify the distribution of energy levels for each sub-structure and also having into consideration the kinetic energy.

3.2 Discrete formulation

When a finite element discretization is used, in terms of displacement, for an undamped model, the free vibration problem results in the following eigenproblem:

Find \((\lambda, U) \neq 0\), such that:

\[
KU = \lambda MU
\]
where $K$ and $M$ are the stiffness and mass matrices, respectively, symmetric and positive-definite, of order $N$, the number of d.o.f. $U$ is the eigenvector associated to the eigenvalue $\lambda$. The experimental results are supposed to be known at the element nodal points and that only one portion has been measured.

Let $\Pi$ be the projection operator allowing the extraction of the measured part of the displacements from the column of the generalized ones. The discretized problem of the minimization of the global modified error, and denoting the nodal values of $u$ and $v$ by $U$ and $V$, is now written as (compare with Prob. (7)):

\[
\begin{align*}
E^2(U, V) &= \|U - V\|^2 + \frac{r}{1-r} \|\Pi U - \Pi V\|^2 \\
= (U - V)^T K(U - V) + \frac{r}{1-r} (\Pi U - \Pi V)^T K_r (\Pi U - \Pi V)
\end{align*}
\]

under the constraint $KU = \lambda MV$

where $K_r$ is a reduced matrix, of order $q \ll N$, $q$ being the number of measured eigenvalues and eigenvectors. In most cases, one applies a Guyan reduction \[41\] to $K$ in order to obtain $K_r$, but other techniques are also possible \[5\].

### 4.1 Continuum mechanics formulation

Now, in the continuum mechanics formulation, we shall have displacement and stress terms, but not the FRFs directly. Those will only be introduced in the discrete formulation. As before, the formalism in the continuum allows for a better understanding and justification of the whole problem.

**Finding the displacement and stress fields**

Find the displacement fields $(U', V') k$ belonging to $V$, minimizing:

\[
E^2(U, V) = \|U - V\|^2 + \frac{r}{1-r} \|\Pi U - \Pi V\|^2
\]

\[
= (U - V)^T K(U - V) + \frac{r}{1-r} (\Pi U - \Pi V)^T K_r (\Pi U - \Pi V)
\]

under the constraint $KU = \lambda MV$

\[
KU = \lambda MV
\]

where $K_r$ is a reduced matrix, of order $q \ll N$, $q$ being the number of measured eigenvalues and eigenvectors. In most cases, one applies a Guyan reduction \[41\] to $K$ in order to obtain $K_r$, but other techniques are also possible \[5\].

### 3.3 Correction process

In Ref. \[4\], a method for parametric correction of the errors detected in stiffness has been proposed. If one defines $K_0$ and $M_0$ as the initial matrices in the finite element model, the problem of finding the design parameters $p_i$ describing the stiffness matrix, is written as follows:

\[
\begin{align*}
E^2(U, V, p) &= \sum_{i=1}^{q \text{ modes}} \left( (U_k - V_k)^T K_0 (U_k - V_k) + \frac{r}{1-r} (\Pi U_k - \Pi V_k)^T K_r (\Pi U_k - \Pi V_k) \right) \\
&\text{subject to } K_0 + \Delta K(p) U_k = \lambda_k M_0 V_k \\
&\text{where } K(p) = K_0 + \Delta K(p) \text{ with } \Delta K(p) = \sum_{i=1}^{q \text{ modes}} c_i p_i \\
&\text{and verifying the physical properties of } K(p) \text{ if } (p_i, c_i, \lambda_i) > 0
\end{align*}
\]

The equilibrium equation in (13) can be written as:

\[
\int_{\Omega} \left[ \text{div} \sigma + \rho \frac{d^2 \bar{u}}{dt^2} \right] \bar{u} \, d\Omega + \int_{\partial \Omega} (F_{\bar{u}} - \bar{\sigma} n) \bar{u} \, dS = 0
\]

Therefore, the local equations to be verified, are:

\[
\begin{align*}
\text{div} \sigma &= \rho \frac{d^2 \bar{u}}{dt^2} \quad \text{in } \Omega \\
\bar{\sigma} n &= F_{\bar{u}} \\
\bar{u} &= u_d
\end{align*}
\]

Let us suppose that the applied force $F_{\bar{u}}$ has been obtained experimentally. Considering the admissible fields $(\bar{u}, \bar{\sigma})$ associated with that force, there will be an error in the constitutive relation, whose expression is the same as in (4), or as in (5), for a description in terms of a displacement field $u$ associated to the stress $\sigma$. For this last case, the
dynamic equilibrium equation becomes:

\[- \int_{\Omega} \delta \epsilon \left( u \right) \varepsilon \left( u^* \right) d\Omega + \int_{\partial \Omega} F_d u^* dS = \int_{\Omega} \rho \frac{d^2 v}{dt^2} u^* d\Omega \]

or

\[ \int_{\Omega} \left( \text{div} \left( \mathbf{H} \epsilon \left( u \right) \right) - \rho \frac{d^2 v}{dt^2} \right) u^* d\Omega + \int_{\partial \Omega} \left( F_d - \mathbf{H} \epsilon \left( u \right) u \right) u^* dS = 0 \]

the local equations being now:

\[ \begin{align*}
\text{div} \left( \mathbf{H} \epsilon \left( u \right) \right) &= \rho \frac{d^2 v}{dt^2} \quad \text{in } \Omega \\
\forall \ t \in [0, T], \ \{ \mathbf{H} \epsilon \left( u \right) \} |_{t=0} &= F_d \\
v_{t=0} &= v_d
\end{align*} \]

Thus, for the localization problem, we have to:

\[ \forall \ t \in [0, T], \ \text{find } \left( u^*, v^* \right), \ \text{minimizing:} \]

\[ \| u - v \|^2 = \int_{\Omega} \varepsilon \left( u - v \right) \varepsilon \left( u - v \right) d\Omega \]

under the constraints

\[ (i) \quad \forall \ u^* \in U \quad \forall \ t \in [0, T] \]

\[ \begin{align*}
- \int_{\Omega} \delta \epsilon \left( u \right) \varepsilon \left( u^* \right) d\Omega + \int_{\partial \Omega} F_d u^* dS = \int_{\Omega} \rho \frac{d^2 v}{dt^2} u^* d\Omega \\
\{ v(x,t_o) \} - v_0 &= \hat{v}_0 \\
v(x,t_o) &= \hat{v}_0
\end{align*} \]

(ii) \[ v_{t=0} = v_d \]

For the same reasons as in section 3.1 we shall consider a modified functional to be minimized. Supposing \( \mathbf{P}_u \) the known part of the measured displacements (admitting, for the moment, that we have got them), the new functional in (19) will be:

\[ \mathcal{E}^2 (u,v) = \| u - v \|^2 + \frac{r}{1-r} \| \mathbf{P}_u - \mathbf{P}_u \|^2 \]

### 4.1 Error indicators

In section 3.1.1, expression (9) is used. Now, instead of taking into account the displacement level contribution for each mode, one should consider such a contribution for the entire frequency range. Therefore, the relative error for each substructure must be given by:

\[ \Gamma_a = \int_{\omega_{a_{\text{max}}}^{\omega_{a_{\text{min}}}}} \left[ \frac{\| u - v \|^2}{\frac{1}{2} \left( \| u \|^2 + \| v \|^2 \right)} \right] d\omega \]

An alternative expression is the following one:

\[ \Gamma_a = \int_{\omega_{a_{\text{max}}}^{\omega_{a_{\text{min}}}}} \left[ \frac{\| u - v \|^2}{\frac{1}{2} \left( \| u \|^2 + \| v \|^2 \right)} \right] d\omega \]

### 4.2 Discrete formulation

When a structure is discretized, the equilibrium equation corresponding to equation (14) is:

\[ M \frac{d^2 \ddot{u}}{dt^2} + K \ddot{u} = f \]

For an harmonic force of complex amplitude \( \hat{f} \) and frequency \( \omega \), the complex amplitude of the response in permanent regime is:

\[ \hat{U} = \left( K - \omega^2 M \right)^{-1} \hat{f} \]

In practice, \( \hat{f} \) and \( \hat{U} \) will be functions of the frequency \( \omega \) and we can write:

\[ \hat{U}(\omega) = \alpha(\omega) \hat{f}(\omega) \]

where \( \alpha(\omega) \) is the FRF matrix of the discretized structure, which - in this case - is the receptance matrix. Each element of \( \alpha(\omega) \), \( \alpha_{ij}(\omega) \), represents the relation between \( \hat{U}_i(\omega) \) (at coordinate \( i \)) and \( \hat{f}_j(\omega) \) (at coordinate \( j \)):

\[ \alpha_{ij}(\omega) = \frac{\hat{U}_i(\omega)}{\hat{f}_j(\omega)} \]

Using (24) and (25), one can write:

\[ \omega^2 \hat{M} \hat{U}(\omega) = \left( K - \alpha^{-1}(\omega) \right) \hat{U}(\omega) \]

Let us suppose \( \alpha(\omega) \) experimentally obtained \( (\hat{g}(\omega)) \) and that those results are reliable ones. Due to errors in the model (in \( K \), in \( M \), or in both), there will be an error in the constitutive relation, whose expression (equivalent to the one in Problem (19)) is:

\[ \| \hat{U}(\omega) - V(\omega) \|^2 = \left( \hat{U}(\omega) - V(\omega) \right)^T K(\hat{U}(\omega) - V(\omega)) \]

with \( \hat{U}(\omega) \) and \( V(\omega) \) verifying

\[ \omega^2 M V(\omega) = \left( K - \hat{g}^{-1}(\omega) \right) V(\omega) \]

We must stress here that the force is expressed through an association between \( \hat{g}^{-1}(\omega) \) and the displacement field \( V(\omega) \). In principle, one could have written the force as \( \hat{g}^{-1}(\omega) V(\omega) \). However, from the formulation in the continuum (Eq. (17)), we see that \( F_d \) is defined over \( \partial \Omega \).
the boundary partition where the forces are imposed, therefore corresponding to the field u. As a consequence, the force is defined in terms of the corresponding discretized field U(ω), i.e., \( g^{-1}(ω) U(ω) \) as in (29).

As, in general, it is not possible to obtain the complete matrix \( g(ω) \), we shall take an hybrid one, \( \alpha(ω) \), composed by terms from the initial finite element model and by experimental values at the measured coordinates. Therefore, Eq. (29) becomes:

\[
\omega^2 M V(ω) = (K - \alpha(ω)^{-1}) U(ω)
\]

and the problem is:

For each \( ω \), find \( (U'(ω), V'(ω)) \) minimizing

\[
\|U(ω) - V(ω)\|^2 = (U(ω) - V(ω))^T K (U(ω) - V(ω))
\]

under the constraint

\[
\omega^2 M V(ω) = (K - \alpha(ω)^{-1}) U(ω)
\]

It is supposed here that the errors in the model are only in the stiffness matrix. It can be shown that the error in the constitutive relation is a quadratic function of the error in \( K(ΔK) \) and that, as we approach the solution, \( \|U(ω) - V(ω)\|^2 \rightarrow 0 \Rightarrow ΔK \rightarrow 0 \).

As before, to avoid the solution \((0, 0)\), we shall work with a modified error functional:

\[
F'(ω) = \|U'(ω) - V'(ω)\|^2 + \frac{r}{1 - r} \|\Pi U'(ω) - \Pi V'(ω)\|^2
\]

(32)

Usually, we do not know \( U(ω) \), but \( \alpha(ω) \). In the common case of only one force being applied at \( j \):

\[
f(ω) = (0,0,...,f_j(ω),...0)^T
\]

If all the d.o.f have been measured,

\[
U_j(ω) = \alpha_j(ω) f_j(ω) \quad i = 1,...,N
\]

(34)

We shall know a complete column of measured FRFs (and, by symmetry, the corresponding row); \( \alpha(ω) \) will look like:

![Diagram](image)

Dividing (34) by the scalar \( U_j(ω) = \alpha_j(ω) f_j(ω) \), it follows that:

\[
\frac{U_j(ω)}{U_j(ω)} = \frac{\alpha_j(ω)}{\alpha_j(ω)} \quad i = 1,...,N
\]

(35)

Therefore, dividing (32) by \( U_j(ω) \), we shall obtain the relations among the FRFs, the other terms just becoming scaled by \( U_j(ω) \):

\[
F'(ω) = \left\| \frac{U(ω)}{U_j(ω)} - \frac{V(ω)}{U_j(ω)} \right\|^2 + \frac{r}{1 - r} \left\| \frac{\Pi U'(ω)}{U_j(ω)} - \frac{\Pi V'(ω)}{U_j(ω)} \right\|^2
\]

(37)

Thus, the problem now is:

For each \( ω \), find \( (U'(ω), V'(ω)) \), minimizing

\[
F'(ω) = \left\| U'(ω) - V'(ω) \right\|^2 + \frac{r}{1 - r} \left\| \Pi U'(ω) - \Pi V'(ω) \right\|^2
\]

under the constraint

\[
\omega^2 M V'(ω) = (K - \alpha(ω)^{-1}) U'(ω)
\]

(38)

To describe \( U'(ω) \) and \( V'(ω) \), a modal basis is used:

\[
U'(ω) = \Psi a(ω)
\]

\[
V'(ω) = \Psi b(ω)
\]

(39)

where \( \Psi \) is the analytical, mass-normalized modal matrix. After some calculations, the functional in (38) becomes:

\[
F'(ω) = (a(ω) - b(ω))^T \omega^2 (a(ω) - b(ω))
\]

\[
+ \frac{r}{1 - r} \left( \Pi \Psi a(ω) - \Pi \Psi b(ω) \right)^T K_r \left( \Pi \Psi a(ω) - \Pi \Psi b(ω) \right)
\]

(40)

where \( K_r \) is a condensed stiffness matrix (although in the examples we have proceeded otherwise - see section 6 for comments on this subject). The equilibrium equation is now written as:

\[
\omega^2 M \Psi b(ω) = (K - \alpha(ω)^{-1}) \Psi a(ω)
\]

(41)

Pre-multiplying (41) by \( \Psi^T \), and after some manipulations, we obtain:

\[
a(ω) - b(ω) = \left[ 1 - \frac{1}{\omega^2} (ω^2 - \Psi^T \alpha(ω)^{-1} \Psi) \right] a(ω)
\]

\[
= R(ω) a(ω)
\]

(42)

Substituting (42) into (40) and making \( \frac{\partial F'(ω)}{\partial m(ω)} = 0 \), it follows that:

\[
R(ω) \omega^2 R(ω) a(ω) + \frac{r}{1 - r} \left[ \Psi^T \Pi^T K_r \Psi a(ω) - \Psi^T \Pi^T K_r \Pi U'(ω) \right] = 0
\]

(43)

from which \( a(ω) \) is calculated, \( b(ω) \) may be calculated from (42), and \( U'(ω) \) and \( V'(ω) \) from (39).

4.2.1 Localization of the wrongly modeled zones

Here, we shall use, as error indicator, the expression
corresponding to equation (22). For a finite element discretization, \( U'(\omega) \) and \( V'(\omega) \) are evaluated for each element, where the contribution of all frequencies is taken into account:

$$
\eta_{el}^2 = \sum_{\omega_{\min}}^{\omega_{\max}} \frac{1}{2} \left[ \left\| U'(\omega) - V'(\omega) \right\|_{L_2}^2 + \left\| V'(\omega) \right\|_{L_2}^2 \right]
$$

(44)

4.2.2 Correction of the model

Although this article is fundamentally about error localization, we present here, briefly, the reasoning for the correction of stiffness errors, once their localization is known. It is an iterative procedure. At iteration 0, the equilibrium equation is:

$$
\omega^2 M V'(\omega) = (K_0 - \alpha_{b_1}^{-1}(\omega)) U'(\omega)
$$

(45)

Introducing a correction \( \Delta K_i \) in (45),

$$
\omega^2 M V_i'(\omega) = (K_0 + \Delta K_i - \alpha_{b_1}^{-1}(\omega)) U_i'(\omega)
$$

(46)

It is necessary to evaluate \( \Delta K_i \). With \( \Delta K_1 \), the new \( K \) is \( K_i = K_0 + \Delta K_i \), and \( \alpha_{b_1}^{-1}(\omega) \) is actualized, becoming \( \alpha_{b_1}^{-1}(\omega) \). In general, at iteration \( i \),

$$
\omega^2 M V_i'(\omega) = (K_{i-1} + \Delta K_i - \alpha_{b_1}^{-1}(\omega)) U_i'(\omega)
$$

(47)

\( \alpha_{b_1}^{-1}(\omega) \) being always actualized in its theoretical part, by solving \( (K_{i-1} - \omega^2 M) \Psi_{i-1} = 0 \), with \( K_i = K_{i-1} + \Delta K_i \).

Writing \( \Delta K_i \) as a function of the design parameters \( p_{ij} \), \( j = 1, \ldots, L \), such that \( \Delta K_i(p_{ij}) = \sum_{j=1}^{L} C_j p_{ij} \), the problem to solve is the following one:

Find, at each iteration \( i \), \( (U'(\omega), V'(\omega)) \) and the design parameters \( p_{ij} \), \( j = 1, \ldots, L \), minimizing

$$
\mathcal{F}_i(\omega) = \sum_{\omega} \left( \left\| U_i'(\omega) - V_i'(\omega) \right\|_{L_2}^2 + \frac{r}{1 - r} \left\| \Psi_i'(\omega) - \Psi_{i-1}'(\omega) \right\|_{L_2}^2 \right)
$$

(48)

under the constraints:

$$
\omega^2 M V_i'(\omega) = (K_{i-1} + \Delta K_i(p_{ij}) - \alpha_{b_1}^{-1}(\omega)) U_i'(\omega)
$$

and \( K_i(p_{ij} = K_i(p_{ij}) + \Delta K_i(p_{ij}) \) with \( \Delta K_i(p_{ij}) = \sum_{j=1}^{L} C_j p_{ij} \)

\( C_j \) : matrix of constant coefficients, deriving from the elementary matrix;

\( p_{ij} \) : parameters of the system at the previous iteration.

Representing \( U_i'(\omega) \) and \( V_i'(\omega) \) through the modal base \( \Psi_{i-1} \):

$$
U_i'(\omega) = \Psi_{i-1}{a_i'(\omega)}
$$

$$
V_i'(\omega) = \Psi_{i-1}{b_i'(\omega)}
$$

(49)

Substituting (49) into (47), pre-multiplying by \( \Psi_{i-1}' \) and after some manipulations, one arrives at:

$$
a_i'(\omega) - b_i'(\omega) = \left[ 1 - \frac{1}{\omega^2} \right] \left( \omega^2 \alpha_{b_1}^{-1}(\omega) \Psi_{i-1} \right) a_i'(\omega) = R'(\omega, p_i) a_i'(\omega)
$$

(50)

Hence,

$$
\mathcal{F}_i(\omega) = \sum_{\omega} \left( a_i'(\omega)^T R'(\omega, p_i)^T \omega^2 \alpha_{b_1}^{-1}(\omega) R'(\omega, p_i) a_i'(\omega) + \left( \Psi_{i-1}' a_i'(\omega) - \Psi_{i-1}' a_i'(\omega) \right)^T K_{r_{i-1}}(\Psi_{i-1}' a_i'(\omega) - \Psi_{i-1}' a_i'(\omega)) \right)
$$

(51)

\( K_{r_{i-1}} \) is the reduced matrix at iteration \( i-1 \). Minimizing the functional, \( \frac{\partial \mathcal{F}_i(\omega)}{\partial a_i'(\omega)} = 0 \), we calculate \( a_i'(\omega) = f(p_i) \).

Substituting back in (51), we obtain the error in terms of \( p_i \). The minimization of this last error expression enables the evaluation of \( p_i \) from which \( K_i \) may be derived.

After a new localization stage for the new value of \( K_i \), the correction stage restarts until convergence is attained.

5 EXAMPLES

The localization method presented in section 4 has been programmed and some examples will now be shown. They are based on a cantilever beam (Fig. 2), discretized into fifteen beam-bar type elements (6 d.o.f each), with the following properties:

- \( S = 10^4 \) m²
- \( l = 8.333332 \times 10^{-11} \) m³
- \( p = 7930 \) Kg/m’
- \( E = 2 \times 10^{11} \) N/m²
- \( L = 0.12 \) m

The total number of d.o.f is 45. The elementary stiffness matrix, \( K_{ij} \), is uncoupled in traction-bending and the elementary mass matrix, \( M_{ij} \), is not diagonal, having rotational inertia terms. There is a vertical force, dynamically applied at d.o.f 32.

The first 10 natural frequencies of the model, in Hertz, are (Tab. 1):

![Fig. 2](image-url)
We shall present 4 examples, where the experimental results are simulated, introducing known inertial errors in some elements, to test the ability of the method.

Example 1
In some elements, we are going to decrease the inertia by around 50% and we shall suppose that all the d.o.f have been measured, with the force always at the same point, i.e., we know a complete column of the receptance matrix. Let us choose the frequency range [1, 200 Hz] (encompassing the first 5 modes), in steps of 20 Hz. We have tested 6 different cases, introducing errors in elements 3, 10, 3+10, 3+4, 3+6 and 10+11. The localization has been perfect in all cases and, just for illustration, we present the last case, for the elements 10 and 11 (Fig. 3).

Example 2
In this example all the d.o.f but the longitudinal ones have been measured, i.e., there are 30 measured d.o.f. When we have tested the same cases as for example 1, the results have been exactly the same. As the decrease in inertia do not affect the longitudinal displacements.

Example 3
In this example, we simulate that the d.o.f No. 3 has not been measured. Here, we started to observe, in some cases, a dependency on the chosen frequency range. For instance, for elements 10 and 11, in the range 1 to 200 Hz, only the element 10 is detected, whereas in the range 1 to 400 Hz, both are well localized (Fig. 4).

Example 4
Considering now that the d.o.f Nos. 3 and 6 have not been measured and always choosing the frequency range 1 to 400 Hz, the results are good. Once again, we present just the case of elements 10 and 11 (Fig. 5).

Other examples
For this model, we have succeeded in taking out the first 3 rotational d.o.f, needing therefore to measure 45 - 15 - 3 = 27 d.o.f. Beyond this point, the method - as it is given here - is no more capable of localizing accurately the errors.

6 DISCUSSION AND CONCLUSIONS
From the many cases tried, we can conclude that:
(i) The proposed functional allows the calculation of well performing error indicators;
(ii) As the number of d.o.f decreases, it is necessary to better reconstruct the experimental information, something which has not been done, as the Guyan reduction has revealed inefficient. Particularly, the loss of information concerning the rotational d.o.f must be compensated. In fact, instead of condensing the stiffness matrix, we have simply replaced the unknown experimental results by the theoretical ones;
(iii) The results show that a single excitation point should be enough;
(iv) The choice of frequencies for the error minimization has been arbitrary, as no particular sensitivity to the proximity of resonances or anti-resonances has been apparent. However, the choice of the frequency range is important, as it must encompass those frequencies where the wrong elements are most deformed.
A possible strategy is to undertake a first calculation of the error over the whole structure and with a wide frequency range and afterwards to select the frequencies giving the greatest values for the errors.

(v) The energy norm chosen for the functional is based upon the deformation energy, but one should not forget that the error tends to increase as we get closer to the free end of the beam and the frequency increases. It is known that the deformation energy decreases as we walk away from the fixed end. The definition of a relative error having also into account the kinetic energy has shown some compensation to that effect.

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