A TUTORIAL ON COMPLEX EIGENVALUES

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ABSTRACT This tutorial examines the nature of natural frequencies and damping ratios for systems with nonproportional damping from first principles. Computational schemes are discussed for the numerical determination of complex eigenvalues. A common misunderstanding of the nature of the natural frequency of a nonproportionally damped system is discussed and illustrated through numerical simulations.

1. INTRODUCTION

This manuscript provides a tutorial on methods of computing the damping ratios and natural frequencies for underdamped mechanical systems with complex eigenvalues. Complex eigenvalues occur when systems have underdamped modes. A system may have underdamped modes, and hence complex eigenvalues and be proportionally damped or nonproportionally damped. Generally, nonproportionally damped systems have complex mode shapes while, proportionally damped systems can be represented with real modes. Less well known is the fact that the undamped natural frequencies determined from complex eigenvalues are different from the natural frequencies calculated from the same system with zero damping. The goal of this manuscript is to examine this difference in "natural frequency" and hence interpret the concept of undamped natural frequency from nonproportionally damped structures as well as to provide a review of calculations for complex eigenvalues.

2. COMPUTATION OF COMPLEX EIGENVALUES

The standard lumped parameter or multiple-degree-of-freedom model of an undamped, autonomous vibrating system is given by the vector differential equation

\[ M \ddot{z}(t) + Kz(t) = 0 \]  (1)

where \( t \) denotes the time; \( M \) is an \( n \times n \) mass matrix, \( K \) is an \( n \times n \) stiffness matrix and \( z \) is an \( n \times 1 \) vector of displacements, or generalized coordinates, each element of which corresponds to a degree of freedom. The over dots denote time derivatives so that \( \ddot{z} \) represent the \( n \times 1 \) vector of accelerations. Often in vibration analysis it is useful to model energy dissipation, or damping, by including the term \( C \dot{z} \) where \( C \) is an \( n \times n \) matrix of damping coefficients and \( \dot{z} \) is the \( n \times 1 \) vector of velocities. Then equation (1) becomes

\[ M \ddot{z}(t) + C \dot{z}(t) + Kz(t) = 0 \]  (2)

Equation (1) is referred to as the undamped system while equation (2) is referred to as the damped system. A complete discussion of these two models can be found in a variety of texts from introductory [1-4] to advanced [5-8]. Knowledge of equation (1) and (2) and the types of systems they model is assumed at the level of the beginning chapters found in any of ref. [1-4].

Usually, but no always, the matrices \( M, C, \) and \( K \) have special properties. For instance \( M, C \) and \( K \) are generally symmetric and positive definite or at least semi-definite (see [1], [4] or [9] for definitions). In addition, if the structure or machine being modeled is spatially repetitive or periodic in nature, these matrices may be banded or sparsely populated. For problems with a large number of degrees-of-freedom, taking advantage of any special structure of the coefficient matrices \( M, C \) and \( K \) could be essential. The algorithms presented in [8] and available in MATLAB are intended to capitalize on any special structure as needed. Computation of the modal information associated with equations (1) and (2) is presented here.

The mode shapes, natural frequencies and damping ratios, collectively called modal data, form the backbone of much vibration analysis and design. Vibration modal data is however strongly related to the algebraic eigenvalue problems, a discipline that...
has received intense attention from the numerical analysis community [9]. This is extremely fortunate for those needing to calculate modal data, as a large number of highly efficient, simple to use and accurate computer algorithms are now commercially available. This section makes the connection between modal data and the various solutions to the algebraic eigenvalue problem [9].

First consider the undamped system described by equation (1). The modal properties associated with solution of the form \( z(t) = \mathbf{u} e^{j\omega t} \) where \( \omega \) is the natural frequency, \( j = \sqrt{-1} \) and \( \mathbf{u} \) is the mode shape (never the zero vector). This substitution yields the algebraic equation

\[
-M \omega^2 \mathbf{u} + K \mathbf{u} = 0
\]

after dividing by the nonzero scalar \( e^{j\omega t} \). Here the mode shape vector \( \mathbf{u} \) is a nonzero vector of constants to be determined and \( \omega \) is the natural frequency to be determined. Equation (3) can be written in a variety of ways. Simple rearrangement yields

\[
\omega^2 M \mathbf{u} = K \mathbf{u}
\]

There will be one value of \( \omega^2 \) and corresponding mode shape \( \mathbf{u}_i \) for each degree of freedom. This is denoted by indexing the modes, i.e., by writing

\[
\omega_i^2 M \mathbf{u}_i = K \mathbf{u}_i \quad i = 1, 2, \ldots n
\]

where \( \mathbf{u}_i \) is not zero although \( \omega_i \) might be (corresponding to rigid body motion). Equation (5) is exactly the generalized eigenvalue problem usually stated in terms of two general matrices \( A \) and \( B \) as

\[
\lambda_i A \mathbf{u}_i = B \mathbf{u}_i \quad \mathbf{u}_i \neq 0
\]

with the obvious connection that \( \lambda_i = \omega_i^2 \). Here \( \lambda_i \) are called the eigenvalues and \( \mathbf{u}_i \) the eigenvectors. It is known from matrix theory [9] that if \( A \) and \( B \) are symmetric with a positive definite then the eigenvalue \( \lambda_i \) and eigenvectors \( \mathbf{u}_i \) are real valued for \( i = 1, 2, \ldots n \). Thus the numerical solution of the generalized eigenvalue problem yields the squares of the natural frequencies (\( \lambda_i = \omega_i^2 \)) and the mode shapes \( \mathbf{u}_i \) directly. Of course the mode shapes \( \mathbf{u}_i \) can only be determined to within a multiplicative constant as is true for the eigenvectors as well. As a result, the vectors \( \mathbf{u}_i \) are often scaled to satisfy a particular condition such as forcing them to have unit magnitude. Numerical algorithms for the generalized eigenvalue problem are well suited for undamped systems that have symmetric, positive definite (or semi definite) and sparse banded matrix coefficients.

The standard eigenvalue problem is of the form

\[
A \mathbf{z} = \lambda \mathbf{z} \quad , \quad \mathbf{z} \neq 0
\]

where \( A \) is restricted to be square and real valued. The undamped modal equation given in equation (4), can be written in the form of the standard eigenvalue problem in two ways. First consider the inverse of the matrix \( M \) denoted by \( M^{-1} \). The inverse of the nonsingular matrix \( M \) is defined to be the matrix \( M^{-1} \) such that the product \( M^{-1} M = MM^{-1} = I \), the \( n \times n \) identity matrix. If the matrix \( M \) has eigenvalues \( \mu_i \) (all nonzero) and eigenvectors \( \mathbf{v}_i \), its inverse can be calculated from forming the \( n \times n \) matrix \( V = [\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_n] \) with columns consisting of the eigenvectors of the matrix \( M \). Then

\[
M^{-1} = V \text{ diag } \left( \frac{1}{\mu_1}, \frac{1}{\mu_2}, \ldots, \frac{1}{\mu_n} \right) V^T
\]

Multiplying equation (4) from the left by \( M^{-1} \) yields the standard eigenvalue problem

\[
M^{-1} K \mathbf{u} = \omega_i^2 \mathbf{u}_i
\]

where again the eigenvalues \( \lambda_i \) are the squares of the natural frequencies and the eigenvectors are the mode shapes. While there are a larger variety of numerical methods available for solving the standard eigenvalue problem than are available for the generalized eigenvalue problem, equation (9) has last any symmetry or sparseness it may have had to begin with. In addition, if the mechanical system being modeled has a large range of values in the mass of its components, computing the inverse may result in an ill conditioned numerical procedure. To solve some of these problems a matrix “divide” can be used to effectively compute \( M^{-1} K \) by using Gaussian elimination [9]. The most efficient way to compute \( M^{-1} \) is to use the Cholesky factorization of \( M \). In fact, the mass matrix may not have an inverse, as in the case of modeling rotational degrees of freedom in finite elements. In such cases, the generalized eigenvalue problem should be used, or a reduction method must first be employed.

A standard eigenvalue problem may be obtained from equation (4) without losing symmetry by introducing the matrix square root. The square root of the matrix \( M \) is that matrix, denoted by \( M^{\frac{1}{2}} \) such that \( M^{\frac{1}{2}} M^{\frac{1}{2}} = M \). If the matrix \( M \) is symmetric and positive definite its eigenvalues, \( \mu_i \) are all real positive numbers and the inverse of the square root of the matrix \( M \), denoted \( M^{-\frac{1}{2}} \), can be calculated from

\[
M^{-\frac{1}{2}} = V \text{ diag } \left( \frac{1}{\sqrt{\mu_1}}, \frac{1}{\sqrt{\mu_2}}, \ldots, \frac{1}{\sqrt{\mu_n}} \right) V^T
\]
Then the eigenvalue (4) can be related to a symmetric standard eigenvalue problem by making the substitution \( u = M^{-\frac{1}{2}}p \) in equation (4) and multiplying by \( M^{-\frac{1}{2}} \) from the left to get
\[
\omega_i^2 p_i = M^{-\frac{1}{2}} K M^{-\frac{1}{2}} p_i
\]
where the mass normalized stiffness matrix \( \hat{K} = M^{-\frac{1}{2}} K M^{-\frac{1}{2}} \) is symmetric as long as \( K \) and \( M^{-\frac{1}{2}} \) are symmetric. Here the eigenvalues of \( \hat{K} \) will again be the squares of the natural frequencies of the systems. However in this case the eigenvectors are related to the mode by shapes
\[
u_i = M^{-\frac{1}{2}} p_i
\]
where \( \nu_i \) are the mode shapes and \( p_i \) are the eigenvectors.

3. DAMPED SYSTEMS

Next consider the damped system of equation (2). Again, the modal data is computed by assuming a solution of the form \( z(t) = u e^{\lambda t} \), where \( u \) is a vector of constants and \( \lambda \) is a scalar. This yields
\[
(M \lambda^2 + C \lambda + K)u = 0
\]
after dividing by the nonzero scalar \( e^{\lambda t} \). Equation (13) is called the nonlinear eigenvalue problem (because of the \( \lambda^2 \) term) or a matrix polynomial (i.e., a polynomial in \( \lambda \) with matrix coefficients). It is also called a lambda matrix \([lo]\). Equation (13) is quite a bit more difficult to solve and is not in a form often of interest to numerical analysts. Hence several transformations are used to reduce equation (13) to the form of either a general or standard eigenvalue problem. At the outset, note that if \( M, C \) and \( K \) are all symmetric and positive definite, \( \lambda \) and \( u \) may still be complex numbers. In fact in the most common underdamped case, \( \lambda_i \) is complex valued. In equation (13) the vector \( u \) as before is considered to be the mode shape.

First consider the case when each value of \( \lambda_i \) that satisfies equation (13) is a complex number. Such systems are said to be underdamped. In fact it is shown in ref. [7], [11] that if \( M, C \) and \( K \) are symmetric and at least positive semi definite than the system is underdamped, i.e., each \( \lambda_i \) is complex valued, if and only if the matrix \( 4K - \hat{C}^2 \) is positive definite where \( \hat{C} = M^{-\frac{1}{2}} C M^{-\frac{1}{2}} \) and \( \hat{K} \) is as defined above. In this circumstance the relationship between \( \lambda_i, \), the natural frequency \( \omega_i \), and the damping ratio \( \zeta_i \) can be defined as follows. Let \( \lambda_i, u_i \) satisfy
\[
(\lambda_i^2 M + \lambda_i C + K)u_i = 0
\]
where \( 4K - \hat{C}^2 \) is positive definite. Then there are \( 2n \) values of \( \lambda_i \) occurring in complex conjugate pairs of the form
\[
\lambda_i = -\zeta_i \omega_i - j \omega_i \sqrt{1 - \zeta_i^2}
\]
where \( j = \sqrt{-1} \), \( \zeta_i \) is the \( i \)th modal damping ratio and \( \omega_i \) is the \( i \)th undamped natural frequency. In particular, let \( \lambda_i \) be complex valued of the form
\[
\lambda_i = \alpha_i + j \beta_i
\]
where \( \alpha_i = \text{Re} \lambda_i \) and \( \beta_i = \text{Im} \lambda_i \). Then
\[
\omega_i = \sqrt{\alpha_i^2 + \beta_i^2}
\]
and
\[
\zeta_i = \frac{-\alpha_i}{\sqrt{\alpha_i^2 + \beta_i^2}}
\]
In this case the eigenvector \( u_i \) also appears in complex conjugate pairs and hence are referred to as complex modes. The analogy given in equations (16) and (17) is used almost exclusively, however \( \omega_i \) given by equation (16) are the same as the frequencies calculated by equation (6) if and only if the matrix \( KM^{-1}C \) is symmetric. This case is called proportional damping and greatly simplifies the computation of the modal data as the complex modes \( u_i \) satisfying equation (13) collapse to the real valued modes of equation (6).

Equation (13) may be related to the generalized eigenvalue problem by defining the vector \( y = u \) and the vector \( y_2 = \lambda u \). Then rewrite equation (13) as
\[
\lambda M y_2 + Cy_2 + Ky_1 = 0
\]
Rearranging equation (18) and writing it below the definition of \( y_2 \) multiplied by the matrix \( K \) yields
\[
\lambda K y_1 = K y_2
\]
\[
-\lambda M y_2 = E y_2 + K y_1
\]
Equation (19) can be written as the partitioned matrix equation
\[
\lambda \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & K \\ K & C \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\]
Now define the \( 2n \times 1 \) vector \( y \) by
\[
y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} u \\ \lambda u \end{bmatrix}
\]
and the \(2n \times 2n\) matrices \(A\) and \(B\) by
\[
A = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}, \quad B = \begin{bmatrix} 0 & K \\ K & C \end{bmatrix} \tag{22}
\]

Equation (20) now becomes the generalized eigenvalue problem (indexing to indicate all the solutions)
\[
\lambda_i A y_i = B y_i; \quad i = 1, 2, \ldots, 2n \tag{23}
\]
which is symmetric if \( M, C \) and \( K \) are all symmetric. Here the natural frequencies and damping ratios are determined by the eigenvalues obtained from the numerical solution of equation (23) by using equations (16) and (17) and the mode shapes are determined from the first \( n \) elements of the vector \( y_i \), i.e., the vector \( u_i \), as indicated in equation (21).

If the matrix \( M \) is nonsingular and has a computable inverse, then equation (14) can be written as (indexing to denote the \( 2n \) solutions)
\[
(\lambda_i^2 I + M^{-1} C \lambda_i + M^{-1} K) u_i = 0; \quad i = 1, 2, \ldots, 2n \tag{24}
\]
Repeating the above steps, i.e., let \( u_i = (y_i)_i \) and defining \( (y_i)_i = \lambda_i u_i \), yields the partitioned form
\[
\lambda \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u_i \\ \lambda_i u_i \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1} K & -M^{-1} C \end{bmatrix} \begin{bmatrix} u_i \\ \lambda_i u_i \end{bmatrix} \tag{25}
\]
where \( I \) is the \( n \times n \) identity matrix and 0 denotes the \( n \times n \) matrix of zeros. Defining
\[
y_i = \begin{bmatrix} u_i \\ \lambda_i u_i \end{bmatrix} \tag{26}
\]
this becomes the standard eigenvalue problem
\[
A y_i = \lambda_i y_i \tag{27}
\]
where the matrix \( A \) called the state matrix takes the form
\[
A = \begin{bmatrix} 0 & I \\ -M^{-1} K & -M^{-1} C \end{bmatrix} \cdot \tag{28}
\]
This standard eigenvalue problem again produces natural frequencies and damping ratios from the solution of equation (27) by use of the formulas given in equation (16) and (17) with mode shapes given by the first \( n \) elements of \( y_i \), listed in equation (26). Note that the above two \( 2n \times 2n \) matrix eigenvalue problems given by equations (23) and (27) give additional alternatives to solving for the undamped natural frequencies by letting \( C = 0 \).

The above mentioned eigenvalue problems can be used to determine modal data. Exactly which of the five approaches to use depends entirely on the nature of a specific problem. In choosing a method, it depends on how important damping is, how spread out the values of the individual masses are, etc.

4. THE UNDAMPED NATURAL FREQUENCIES FOR COMPLEX EIGENVALUES

The common perception in most test book explanations of natural frequency is that \( \omega_i \) calculated by equation (16) are the undamped natural frequencies and hence must also be equal to the value for \( \omega_i \) calculated in equations (5), (9) or (11). That is, the undamped natural frequency calculated for \( C = 0 \) are often mistakenly assumed to be the same as those calculated for \( C \neq 0 \) by equation (16). This is false, except in those cases where the damping is such that the equations of motion completely decouple. Such systems are often called: proportionally damped systems, normal mode systems or classically damped systems.

Mathematically, a necessary or sufficient condition for the equation of motion to decouple is that the matrix \( CM^{-1}K \) is symmetric. One common case where this is true is of course proportionally damped systems where \( C = \alpha M + \beta K \) (\( \alpha, \beta \) constant). For such systems the real valued eigenvectors of equations (11) and (12) can also be used for the eigenvectors of equation (14) and the frequency \( \omega_i \) calculated from equation (16) are exactly equal to those calculated by equation (5). If however, \( CM^{-1}K \) is not symmetric the equations of motion do not decouple and the two calculations lead to different values as the following numerical example illustrates.

Example: The example of this section is a simple two degree of freedom system that can be used to examine the effects of coupling by looking at a simple scalar parameter. Let \( M = I \), the two by two identity matrix and let
\[
\begin{bmatrix} M & K \\ K & C \end{bmatrix}
\]
and note that \( Y \) defines the “nonproportional” part of the damping matrix. For \( C = 0 \), this system is proportionally damped. As \( Y \) increases the matrix \( CM^{-1}K \) becomes asymmetric and proportional damping is lost. The undamped natural frequencies are easily calculated for \( C = 0 \) to be
\[
\omega_1 = 0.8740 \text{ rad/sec} \\
\omega_2 = 2.2882 \text{ rad/sec}
\]
from equation (11). Table 1 illustrates values of $\omega_i$ for this system calculated using equation (16) as the coupling factor $\nu$ is increased from zero to one. The damping ratio is also listed, so that it is clear this is not a function of heavy damping, but rather a function of geometric coupling.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\omega_{1,2}$ from (16)</th>
<th>$\zeta_{1,2}$ from (17)</th>
<th>$\Delta \omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.8740</td>
<td>0.1009</td>
<td>0 %</td>
</tr>
<tr>
<td></td>
<td>2.2862</td>
<td>0.1363</td>
<td>0 %</td>
</tr>
<tr>
<td>0.25</td>
<td>0.8790</td>
<td>0.1656</td>
<td>0.5 %</td>
</tr>
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<td></td>
<td>2.2752</td>
<td>0.1118</td>
<td>2.3 %</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8944</td>
<td>0.2342</td>
<td>2 %</td>
</tr>
<tr>
<td></td>
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<td>0.0852</td>
<td>3.3 %</td>
</tr>
<tr>
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<td>0.9205</td>
<td>0.3116</td>
<td>5.3 %</td>
</tr>
<tr>
<td></td>
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<td>0.0521</td>
<td>5 %</td>
</tr>
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<td>0.4033</td>
<td>9.4 %</td>
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<tr>
<td></td>
<td>2.0913</td>
<td>0.0068</td>
<td>8.4 %</td>
</tr>
</tbody>
</table>

TABLE 1
Shift in undamped natural frequency in rad/sec due to coupling caused by damping.

Note from the table, that even in cases of low damping ratio in one mode, a very large difference in the undamped natural frequency calculation results from the complex eigenvalue versus the undamped system. In some cases the difference between the undamped natural frequency as calculated from the damped structure is in error more than 9% from the actual undamped natural frequency calculated directly from $M$ and $K$. The results of the frequency shift illustrated in Table 1 are due only to the coupling of the equations of motion caused by the damping parameter $\nu$ and are not to be confused with the damped natural frequencies. For instance, the first damped natural frequency for the heavily coupled case $\nu = 1$, is $\omega_d = \omega_1 \sqrt{1 - \zeta_1^2} = 0.8752$ rad/sec, whereas $\omega_1$ calculated from the stiffness and mass matrix only is $\omega_1 = 0.8740$ rad/sec and $\omega_1$ calculated from the damped system using equation (16) is $\omega_1 = 0.9564$ rad/sec.

$\omega_{dr}$ denotes the driving frequency. The steady state response is given in [7,10] to be

$$x(t) = \sum_{i=1}^{4} \text{Re} \left\{ e^{-\lambda_i(t-\tau)} u_i u_i^T \right\} f(\tau) d\tau$$

where $u_i$ are the complex eigenvectors defined by equation (24) and the applied force is harmonic, i.e.,

$$f(\tau) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin \omega_{dr} \tau$$

The complex valued eigenvectors are normalized by the rule

$$u_i^T (2M \lambda_i + D) u_i = 1$$

This sum will give rise to terms with amplitudes of the form

$$\frac{1}{\sqrt{(\omega_i^2 - \omega_{dr}^2)^2 + (2\zeta_i \omega_d \omega_i)^2}}$$

where $\omega_i$ is the value calculated from equation (16). Thus resonance will occur in the example problem near $\omega_2 = 2.0913$ rad/sec and not $\omega = 2.2882$ rad/sec as predicted by the undamped system, even though the damping ratio for that mode is only 0.68%.

6. SUMMARY
Several methods for computing eigenvalues and subsequent modal data have been reviewed. The most numerically efficient and accurate method to use is computing frequencies and damping ratios depends and the structure of the $M$, $C$ and $K$ matrices as well as these relative component values. It has been pointed out that the undamped natural frequency for a nonproportionally damped system can be quite different than the undamped natural frequency calculated assuming zero damping.

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