A TEST-ANALYSIS-MATRIX (TAM) WITH IMPROVED DYNAMIC CHARACTERISTICS

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ABSTRACT

Whenever experimentally derived (Test) modal vectors are compared to the analytically obtained (Analysis) modal vectors, for purpose of model verification or model updating, a transformation matrix must be utilized to expand/reduce the degrees of freedom, DOF, so that the spatial dimension of the two sets of modal vectors is the same. The measured set of DOF seldom match the analytical set of DOF in dimension. Several Test Analysis Matrix (TAM) approaches such as the Guyan reduction, Improved Reduced System, System Equivalent Reduction/Expansion Process and the Hybrid TAM have been developed to achieve such a transformation. One of the simplest examples of utilizing such a TAM is the computation of an orthogonal matrix involving the modal test vectors, the analytical mass matrix and the analytical vectors. An important requirement of this correlation is that the reduced matrices of mass and stiffness represent the same dynamics as the actual physical structure. Again the TAMs need to incorporate the target mode dynamics and the equally important residual dynamics. The present investigation aims at developing a new transformation mapping that reduces/expands the analytical vectors to the required size of the experimental DOF for such an orthogonality check with effective residual flexibility. The residual subspace used in this mapping has, as its basis, modes derived using the residual flexibility matrix that carry with them the effect of all of the residual set of normal modes. Simulated test-analysis correlation results are represented by FEM target modes with noise modeled as a random linear combination of all FE 44 modes.

Nomenclature

\[ M, K, C, G : \] mass, stiffness, damping and flexibility matrices of analytical model.
\[ \Psi, \Phi : \] modal matrix of modes stacked as columns.
\[ \mathbf{u} : \] displacement vector.
\[ X(w), F(w) : \] displacement and force vectors.
\[ \mathbf{q} : \] modal coordinates.
\[ \lambda : \] modal frequency.
\[ \sigma, \omega : \] damping factor and natural frequency.
\[ a, o, n : \] active, omitted and total DOFs.
\[ t, r : \] subscripts representing target and residual modes.

Introduction

Accurate analytical models of structures are required for a variety of reasons. These may be for model verification/updating, failure analysis, control system design or design modification requirement. The analytical model is usually based upon a Finite Element Model, FEM. Accuracy is often assessed by comparing the modal characteristics extracted using vibration test data with the corresponding modal properties predicted by the analytical model.

Test and analysis frequencies can be compared directly if the corresponding modal vectors can be identified. However modal vectors cannot be readily compared because of the large difference in the number of analytical
and test degrees of freedom, DOF. In practice, the
dimension of the analytical DOF set, that more completely
describes the structure, is greater than that of test DOF set.
A direct comparison is possible if the test vectors are expanded to the full analytical model space or the analytical vectors are reduced using a transformation or mapping to the space of the test DOF. The latter results in reduced mass and stiffness matrices in only the test DOF. It is also generally regarded that the mass matrix represents the inertia of the system more accurately than the stiffness matrix models the elasticity of the system. For this reason the mass matrix at the reduced space is often used as a weighting matrix in test-analysis mode shape correlation computations such as orthogonality and cross orthogonality and, in this work, this approach is followed.

In order to obtain meaningful conclusions from a test-analysis correlation using a Test-Analysis-Matrix, TAM, approach, the TAM must accurately represent the physical dynamics of the structure. This implication is twofold

1. The FEM must model the dynamics of the structure as accurately as possible, and,

2. The subsequent TAM used in correlation should adequately reflect the FEM dynamics. In practice a TAM is sought to validate the test results against the analytical results but the TAM in itself depends greatly on the FE Model. In reality, existing (and proven) TAMs are used for the first step in the validation of the FE model of the system. The endeavor now is to construct an improved TAM that is superior in terms of its sensitivity to noise, i.e., the TAM behaves well in reducing/expanding the analytical/test vectors, respectively. In other words, a robust TAM is desired.

The DOFs are divided into two sets, the active set which are the measured DOF for a structure and those that are not measured but comprise the FE model DOF to adequately model the structure’s dynamic behavior. In the sequel the active DOF shall be denoted by a-DOF and the inactive as the omitted DOF or o-DOF. The total DOF of a system is then the sum of the two and shall be denoted by n-DOF.

A number of TAMs have been developed in the past, the simplest ones based on the Guyan condensation[1]. These TAMs use the static modes (or constraint modes) and as such may reflect the flexibility of the structure in its lower modes to some extent but are quite sensitive to the degrees of freedom chosen as the inactive DOF. This class of TAMs shall be referred to as the static TAM. An improvement to the static TAM is the Improved Reduced System, IRS, presented by O’Callahan[2] that includes a static approximation to the dynamic inertia terms discarded in the Guyan reduction and as such is an improvement over the static TAM. Modal TAMs have subsequently been presented by O’Callahan[3] as the System Equivalent Reduction/Expansion Process, SEREP and by Kammer[4] that use the analytical target modal vectors as the basis of generation of the required transformation. Kammer[5] then developed a Hybrid TAM that combines the Modal TAM and the static TAM in an attempt to represent the residual dynamics. A review and comparison of these TAMs is contained in Ref[6].

The Modal TAM has been successfully applied in test-analysis correlation for several large structures. However it is observed that in some cases the use of modal TAM in test-analysis orthogonality and cross-orthogonality computation can result in larger off-diagonal terms than the corresponding off-diagonal tams produced by a less accurate static TAM. This sensitivity to discrepancies between test and FE model mode shapes is conjectured to be due to the Modal TAM’s poor representation of residual modes. In this paper, residual modes are defined to be those mode shapes which are not used in the generation of the transformation matrix. These are the modes that are not targeted for identification and correlation and generally correspond to frequencies below and/or above the frequency range of interest.

In general, the test modes contain noise. Some sources of noise are transducer and the signal processing noise, the most common being the sensor position and alignment errors as well as the signal processing leakage error. In addition, each test mode will also contain noise due to corruption from other modes. Methods for extracting mode shapes from test data are not perfect: therefore, each extracted test mode from a parameter estimation algorithm could be corrupted by a linear combination of other modes. Many of these corrupting mode shapes will be the residual modes. This will especially be the case for a system with large number of degrees of freedom where due to high modal density, the target modes may be
interspersed with many residual modes. This can also be the case when the target modes are not well excited by the chosen exciter locations. All the residual modes cannot be included due to the inherent nature of the reduction process; the number of modes that can be included in the transformation generation are limited by the number of active DOF or the number of sensors for the test. Thus, the lack of sensors constrain this inclusion of residual modes in the Modal TAM generation within the frequency range of interest. This is discussed more fully with the numerical example illustrating the TAM performance. One of the causes of the sensitivity of the orthogonality checks leading to higher off-diagonal terms can be shown to be due to Modal TAM’s poor representation of these residual modes. Kammer developed a hybrid TAM to reduce this effect.

The static modes may not be able to accurately span all the target modes, but the static modes together with the target modes do span the residual mode set rather effectively. The target modes by themselves cannot span the residual space with which they are linearly independent for a reciprocal system. The static modes may give reasonably accurate predictions for many residual modes, particularly if they are of low order.

In this work, a transformation is developed that preserves the exact EM target mode representation and incorporates the residual mode information. These residual modes are not computed via a regular eigenanalysis but are made available as a linear combination of all the residual eigenvectors, obtained from the flexibility matrix of the system and the computed target modes. In other words, the residual modes are never computed directly. The present investigation aims at determining if such a transformation, that provides an overall representation of the target modes and also all of the residual modes via the above residual subspace, spans the modal space of the structure more closely and is therefore less sensitive than the Modal TAM to residual corruption of the test mode shapes.

2 Theoretical development

The modal vectors from the FE model are mass and stiffness matrix orthogonal, i.e., if $\Psi$ is the modal matrix with modes as its columns, then the modal vectors when mass orthonormalized imply the following

$$\Psi^T M \Psi = [I]$$

$$\Psi^T K \Psi = \text{Diag}(\lambda^2)$$

(1)

The system matrices can be reduced to the size of test DOF if a mapping is available to relate the displacement in the o-set DOF to the a-set DOF. If $T$ is such a transformation then the reduced space matrices are obtained as

$$M_{RED} = T^T M T$$

$$K_{RED} = T^T K T$$

(2)

The superscript $T$ in the above expressions refer to the transpose of the matrix. If the reduced mass matrix accurately represents the inertia of the system, then the test vectors when normalized to a unity modal mass should result in an identity matrix and indicate a perfect orthogonality of the test modes with respect to the mass matrix. An orthogonality check can be performed between the test vectors with the analytical reduced mass matrix as the weighting matrix or a cross-orthogonality between the test vectors and the reduced space analytical modal vectors. In reality, the test vectors are corrupted by noise from various sources as discussed previously. Also the FE model may not model the system perfectly. Therefore, the off-diagonal terms of such an orthogonality will not be 0.0.

In practice, the orthogonality is classified acceptable if the off-diagonal terms are less than 0.10 and the diagonal terms off by $\pm 0.10$ from a value of unity.

In a related comparison, when the test modes are compared to the analytical modes via a mass matrix, the orthogonality check shall be referred to as the cross-orthogonality check and this generalized mass obtained as the cross-modal-mass, CMM.

Typically the number of response points on a structure in a test are fewer than the FE DOF needed to model the structure. If this is so and the test modes, when extracted using a parameter estimation algorithm are normalized to a unity modal mass, then the modal coefficients of the test vector at the response locations match exactly the modal coefficients of the corresponding analytical modal vectors.
for a similar scaling if the analytical model exactly represents the system dynamics (even though the two vectors are not of the same length).

This can be verified by simulating test results. The mass and stiffness matrix of a theoretical structure are generated and the frequency response functions, FRFs of this theoretical structure evaluated by an inversion of the impedance matrix at each frequency in the frequency range of interest. A proportional damping matrix is included to permit this inverse at the natural frequencies of the system:

\[
\{X(\omega)\} = \{-\omega^2 \mathbf{M} + j\omega \mathbf{C} + \mathbf{K}\}^{-1}\{F(\omega)\} \tag{3}
\]

with \(\{F(\omega)\}\) being an incidence vector with unity at the input reference location and zeros at other locations. The corresponding \(\{X(\omega)\}\) are then the FRFs \(\{H(\omega)\}\) of the system with one driving point FRF that corresponds to the input reference. A Polyreference Frequency Domain parameter estimation algorithm can be employed directly on the computed FRFs or a Polyreference Time Domain parameter estimation algorithm can be applied on the impulse response functions, IRFs. The IRFs are computed by taking an inverse Fourier transform of the FRFs.

The number of response points employed in the above parameter estimation process, that also determine the length of the test vector computed, is the response in the simulated active DOF set, a-DOF. These response points can now be varied to simulate different a-set DOF. These modal coefficients exactly correspond to the analytical modal vector coefficients obtained by an eigenanalysis of the mass and stiffness matrix.

This leads to the conclusion that if the analytical model is an exact representation of the actual structure then test vector coefficients at the response locations are exactly the same as the analytical vector for a similar scaling (though the vectors are of different length).

Typical TAMs are next examined briefly from the point of view of expansion/reduction for cross-orthogonality checks. Ref[3] transforms the physical DOFs to the modal coordinates as:

\[
\{u\} = [\Psi]\{p\} \tag{4}
\]

Or in the partitioned form

\[
\begin{align*}
\{u_a\} &= \left[\begin{array}{c}
\Psi_a \\
\Psi_o
\end{array}\right]\{p\} \\
\{u_o\} &= \{p\}
\end{align*} \tag{5}
\]

Writing out Equation (4) and solving for the modal coordinates using a generalized inverse approach yields:

\[
\{p\} = [\Psi_a^T]\{u_a\} \tag{6}
\]

The displacement vector at the full space can then expressed in terms of the active DOFs as:

\[
\{u_n\} = \left[\begin{array}{c}
u_a \\
u_o
\end{array}\right] = \left[\begin{array}{c}
\Psi_a \\
\Psi_o\Psi_a^T
\end{array}\right]\{u_a\} \tag{7}
\]

The transformation \([T]\) matrix in this case is:

\[
[T] = \left[\begin{array}{cc}
\Psi_a \\
\Psi_o\Psi_a^T
\end{array}\right] \tag{8}
\]

where \(T\) is the mapping from a-set to the at (a+o)-set or n-set DOF. Frequently the upper-sub matrix in the above expression is replaced by an identity matrix of the a-set size to prevent any smoothing of test vectors on expansion. The Modal TAM results when only the target modes are included in the modal matrix of Equation(4), the transformation matrix \(T\) in this case is denoted as \(T_m\) in the following. The mass and stiffness matrices in the reduced space or in the test DOF are generated as previously noted:

\[
\begin{align*}
\mathbf{M}_{RED} &= \mathbf{T}^\top \mathbf{M} \mathbf{T} \\
\mathbf{K}_{RED} &= \mathbf{T}^\top \mathbf{K} \mathbf{T}
\end{align*} \tag{9}
\]
The computed eigenvectors of these reduced space matrices on normalization that correspond to the FE target modes in the modal matrix of Equation (4) have the same modal coefficients as the coefficients of original full space eigenvectors for the target modes.

The above two results lead to the conclusion that the test vectors are identical to the analytical reduced space target eigenvectors (when scaled uniformly) if the analytical model is an exact representation of the structure’s dynamics.

The above also leads to the conclusion that an ideal cross-orthogonality between the test and the analytical vectors will result (an identity matrix) at the reduced space if the test vectors are exact (there is no noise) and assuming that the analytical model is tuned to the actual structure. In other words, any discrepancies in the test and reduced space modal vectors can be directly attributed to the Finite Element modeling errors or errors in the test vectors. Therefore, an effective TAM needs to be developed to improve the cross-orthogonality check. The deterioration of orthogonality due to such a contamination has also been pointed by Kammer[5].

Therefore, the natural requirement is to have the analytical model tuned to the physical structure so that the two represent the same dynamics. This leads to the realm of model updating and the purpose of the present work is to evolve an effective TAM.

The FE displacement vector \( u_n \) in terms of the target and residual mode contributions, \( u_{\text{fem}} \) and \( u_{\text{rfem}} \) is:

\[
\begin{align*}
\mathbf{u}_n &= \mathbf{u}_{\text{fem}} + \mathbf{u}_{\text{rfem}} \\
&= \Psi_t \mathbf{q}_t + \Psi_r \mathbf{q}_r 
\end{align*}
\]  

(10)

\( \Psi_t, \Psi_r, \mathbf{q}_t, \mathbf{q}_r \) are the target and residual modal matrices and coordinates, respectively. The number of the target and the residual modes is then the total number of eigen modes of this finite dimensional problem. Again the above displacement vector \( \mathbf{u}_n \) can be partitioned into the a-set and the o-set DOF with the relationship \( n = a + d \). In most cases, the number of active DOFs are greater than the number of target modes, but the test structure could be spatially under sampled too. The number of active and omitted DOFs are denoted by \( n_a \) and \( n_o \) respectively. It can be shown that the Modal TAMs based on target modes alone attempt to span the \( n_o \) dimensional residual o-set space with only \( n_t \) independent vectors that may not even span the o-set residual modal partitions \( \mathbf{\Psi}_{ro} \) effectively leading to a deterioration of orthogonality.

The residual subspace that is a linear combination of all of the residual modes is generated as follows. If \( \mathbf{\Psi} \) is the modal matrix of all the eigenvectors of the system, then they form a complete set for a reciprocal system and

\[
\mathbf{\Psi}^T \mathbf{K} \mathbf{\Psi} = [\text{Diag}(\lambda^2)]
\]

(11)

Partitioning the modal vectors into target and residual sets:

\[
\mathbf{G} = \mathbf{K}^{-1} = [\mathbf{\Psi}_t \mathbf{\Psi}_r] \begin{bmatrix} \lambda_t^2 & 0 \\ 0 & \lambda_r^2 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{\Psi}_t \mathbf{\Psi}_r \end{bmatrix}^T
\]

(12)

Expanding out for the residual modes:

\[
\mathbf{\Psi}_r \begin{bmatrix} \lambda_r^2 \\ 1 \end{bmatrix} \mathbf{\Psi}_r^T = \mathbf{G} - \mathbf{\Psi}_t \begin{bmatrix} \lambda_t^2 \\ 1 \end{bmatrix} \mathbf{\Psi}_t^T
\]

(13)

Thus the residual subspace is available from the flexibility matrix of the system and the computed target modes. The flexibility of the system can normally be made available internally during a FEM Gaussian elimination required in an eigenanalysis. In the following, the transformation matrix analogous to \( \mathbf{T}_m \) computed using this residual space will be denoted by \( \mathbf{T}_r \).

### 2.1 TAM Formulation

A TAM is, therefore, required that retains the exact representation of the Modal TAM for the target modes and at the same time produces the accurate representation of the residual modes. To achieve this, the Modal TAM reduction for the target modes can be examined:

\[
\tilde{\mathbf{u}}_{\text{fem}} = \mathbf{T}_m \mathbf{u}_a = \begin{bmatrix} I \\ \mathbf{R}_{tm} \end{bmatrix} \mathbf{u}_a \text{ with } \mathbf{u}_a
\]

(14)

\[
\mathbf{R}_{tm} = \mathbf{\Psi}_r [\mathbf{\Psi}_t \mathbf{T} \mathbf{\Psi}_t]^{-1} \mathbf{\Psi}_t \mathbf{T}
\]
The $n_a$ linearly independent columns of $T_m$ provide an exact representation of the target modes but a poor representation of the residual dynamics with which they are linearly independent. The reduced space vector $u_a \in \mathbb{R}^a$ where $\mathbb{R}^a$ is an $n_a$ dimensional space containing the dynamics of the reduced space system. The target modes are exactly reduced as noted earlier. The exact target mode dynamics is given by:

$$u_{ta} = T_m u_a = \begin{pmatrix} 1 \\ R_m \\ u_a \end{pmatrix}$$

(15)

where the $u_{ta}$ is a vector of the subspace of $\mathbb{R}^a$ which is spanned by the FE target modes partitions $\Psi_{ta}$. Thus $u_{ta}$ is the restriction of $u_a$ to the column space of $\Psi_{ta}$.

An effective way of combining the target and residual dynamics is to construct them as complimentary subspaces of $\mathbb{R}^a$ and then projecting the reduced space vector separately onto each subspace. This projection is achieved via a projector matrix $P$. A projector matrix is an idempotent matrix $P = P$ which divides the space $\mathbb{R}^a$ into two complimentary subspaces

$$\mathbb{R}^a = C \oplus N$$

(16)

where $C$ is the column space of $P$ and $N$ is the null space of $P$ and $\oplus$ is the direct sum operator. If $P$ is constructed such that $C = \text{col}(\Psi_{ta})$, then $P$ will have a rank of $n_t$, $C$ will be of dimension $n_t$, $N$ will have a dimension of $n_a - n_t$ and the following is a property of this projection matrix.

$$u_{ta} = P u_a$$

(17)

The target mode contribution to the TAM displacement vector $u_{ta}$ is the projection of $u_a$ onto the column space of $\Psi_{ta}$ along the null space. Substituting the above into Equation (15) yields

$$\hat{u}_{fem} = T_m Pu_a$$

(18)

where the matrix product $T_m Pu_a$ results in the following

$$T_m Pu_a = T_m u_a \text{ if } u_a \in C$$

$$\forall 0 \text{ if } u_a \in N$$

(19)

The same concept is used to form the null space $N$ of the matrix $P$. In this way the exact target mode representation is maintained in the proposed TAM together with the residual dynamics. In order to generate $u_{NS}$, an element of the null space of the matrix $P$, the projector matrix $P_N$ is formed via the complement of $P$:

$$P_N = I_a - P$$

(20)

in which $I_a$ is an $n_a$ dimensional identity matrix. Matrix $P_N$ is the projection onto the null space of $P$ along its column space. The null space displacement vector $u_{NS}$ is thus given by:

$$u_{NS} = P_N u_a$$

(21)

The approximation for the residual dynamics is given by:

$$\hat{u}_{fem} = T_r P_N u_a$$

(22)

where matrix $T_r P_N$ is the extension of the restriction of $T_r$ to the null space of $P$. The complete TAM approximation to the FE displacement vector is given by:

$$\hat{u}_{fem} = T_m Pu_a + T_r P_N u_a = [T_m P + T_r P_N] u_a$$

(23)

A similar approach is adopted in Ref[5]. The technique used here in evolving the projector matrix however uses the spaces in a direct manner. If $\Psi_{ta}$ is the ’a’ set partition of the target modes then this set is of rank $t$, where $t$ is the number of target modes assuming that $a > t$. This is the case in practice. Then one of the complimentary subspaces of $\Psi_{ta}$ in $\mathbb{R}^a$ is the null space of $\Psi_{ta}$ transposed. Again the null space of $P$ can be selected as the modes from the residual space and these are denoted here by $\Phi_{ra}$. This is the a-set partition of the residual subspace of Equation (13). The number of modes that can be used to form the null space are restricted to $n_a - n_t$, since $n_t$ is the dimension of columnspace of matrix $P$. Let $0$ be a matrix of zero entries of size $n_a$-by-$n_t$, and following Ref[7], with the following conditions on $P$:

$$P \Psi_{ta} = \Psi_{ta}$$

$$P \Phi_{ra} = 0$$

(24)
Then the single expression, in terms of product of matrix partitions that leads to an evaluation of \( P \), can be written as:

\[
P \left[ \Psi_{ta} \Phi_{ra} \right] = \left[ \Psi_{ta} O \right]
\]  
(25)

Finally:

\[
P = \left[ \Psi_{ta} O \right] \left[ \Psi_{ta} \Phi_{ra} \right]^{-1}
\]  
(26)

It can be verified that \( P^2 = P \) by using Equation (25).

Using relationships similar to those in Equation (9), the modified TAM mass and stiffness matrices are given by

\[
M_m = P^T M_m P + P^T T_m T M_{fem} T_r P N + P N T T_r T M_{fem} T_m P + P N T M_r P N
\]  
(27)

\[
K_m = P^T K_m P + P^T T_m T K_{fem} T_r P N + P N T T_r T K_{fem} T_m P + P N T K_r P N
\]  
(28)

in which \( M_m \) and \( K_m \) are the modal TAM mass and stiffness matrices respectively given by the following expressions and \( M_r \) and \( K_r \) are obtained as

\[
M_m = T^T M_m T \quad \text{and} \quad K_m = T^T K_m T
\]

\[
M_r = T^T M_r T \quad \text{and} \quad K_r = T^T K_r T
\]  
(29)

If Equations (27) and (28) are pre- and post-multiplied by target mode partitions \( \Psi_{ta} \) due to the fact \( PT \Psi_{ta} = \Psi_{ta} \) and \( P N \Psi_{ta} = 0 \), the following expressions result:

\[
\Psi_{ta}^T M_m \Psi_{ta} = \Psi_{ta}^T M_m \Psi_{ta} = I_t
\]  
(30)

where \( I_t \) is a t-by-t identity matrix. Therefore, the modified TAM exactly predicts the target modes and frequencies.

Again if Equations (27) and (28) are pre- and post-multiplied by \( \Phi_{ra} \), the results confirm the residual dynamics:

\[
\Phi_{ra}^T M_m \Phi_{ra} = \Phi_{ra}^T M_r \Phi_{ra}
\]
\[
\Phi_{ra}^T K_m \Phi_{ra} = \Phi_{ra}^T K_r \Phi_{ra}
\]  
(31)

3 Numerical Examples of Test-Analysis-Matrix correlation

Examples are considered to demonstrate the correlation of test and analytical vectors via an orthogonality check using the TAM approach. The model considered is a plane frame structure of 21 nodes. The FE model has 45 beam elements and 57 degrees of freedom. 38 of these are considered the active DOF or the ‘a’-set DOF and the remaining 19 DOF are then the omitted DOF or the ‘o’-set DOF. Thus, in a physical structure this would correspond to 38 sensor locations. All rotational DOF form the omitted DOF in keeping with lack of rotational information in most practical cases. The first ten modes are considered as the target modes and the remaining 47 FE modes are then considered the residual modes.

Modal frequencies obtained from reduced system matrices using the different TAMs are compared to the FEM frequencies. The percentage error is plotted in Fig. 1. It is noted that the static TAM does capture the eigen frequencies to a good extent. This implies that the constraint modes that are in effect a linear combination of the eigenvectors of the stiffness matrix alone ( the inertia of the system discarded ), span the lower modes of the system rather well, with accuracy of span deteriorating with increasing frequency as shown in Table 1 and plotted in Fig. 1. The Modal TAM, on the other hand represents all the 10 target frequencies exactly with a poor job of the residual modes. The reduced space mass and stiffness matrices via a Modal TAM form a rank deficient system if the linear transformation matrix \( T \) has fewer than \( n_a \) modes. The Hybrid TAM[5] that incorporates static
modes with the target modes predicts all the target modes exactly and also the residual frequencies well. The present TAM predicts the target modes exactly and the residual modes reasonably since the linear transformation used has the residual space modes represented as their linear combination. The higher modes are in error as they are weighted very low due to the inverse matrix in Equation (13). Test vectors are simulated by FEM target modes with superposed noise modeled as a random linear combination of all FEM modes.

Mode shape correlation can be represented in terms of the diagonal values of a weighted orthogonality check between test vectors and the reduced space analytical vectors using the TAM mass matrix. This value is referred to as the cross-modal-mass, CMM. An ideal value is a 1.0. The deviation from a value of unity is shown in Fig. 2 for the Present TAM and Hybrid TAM. A rather low deviation occurs with the Present TAM. Again in an ideal situation implying perfect orthogonality, the off-diagonal terms need be 0.0. Therefore, a criteria for comparison is the Root Mean Square, RMS, value taking all the off-diagonal terms. The static TAM has the largest value confirming its approximate dynamic characteristic. The Modal TAM is an improvement and the present TAM is equally good.

As noted in Ref[5], superior orthogonality results when target modes using a Modal TAM are combined in a hybrid manner compared to when the Modal TAM alone is used. The number of residual modes that can be used in a TAM generation is in effect limited by the number of active DOF. The difference in the number of active DOF and the number of target modes is the number of residual modes that can be used in a TAM generation using either method of a TAM generation. One method is the use of projection matrix that divides the R space into two complimentary subspaces and the other method is the direct one where the residual modes are stacked with the target modes in the modal matrix when constructing the generalized inverse. The rank of \( T,P_N \) in Equation (22) is limited by the rank of the null space projector \( P_N \) which is \( n_r \).

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<td>6.2609</td>
<td>6.2027</td>
<td>6.2027</td>
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</tbody>
</table>

As noted in Ref[5], superior orthogonality results when target modes using a Modal TAM are combined in a hybrid manner compared to when the Modal TAM alone is used. The number of residual modes that can be used in a TAM generation is in effect limited by the number of active DOF. The difference in the number of active DOF and the number of target modes is the number of residual modes that can be used in a TAM generation using either method of a TAM generation. One method is the use of projection matrix that divides the R space into two complimentary subspaces and the other method is the direct one where the residual modes are stacked with the target modes in the modal matrix when constructing the generalized inverse. The rank of \( T,P_N \) in Equation (22) is limited by the rank of the null space projector \( P_N \) which is \( n_r \).

If the number of modes are greater than the number of active DOF, then the equations are solved in an average sense as opposed to the least squares solution when the

<table>
<thead>
<tr>
<th>Residual Modes</th>
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<tr>
<td>11</td>
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</tbody>
</table>

Table 1 Comparison of Modal Frequencies using Static, Hybrid and Present TAMs.

<table>
<thead>
<tr>
<th>Static TAM</th>
<th>Hybrid TAM</th>
<th>Present TAM</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00034328</td>
<td>0.00034233</td>
<td>0.00035324</td>
</tr>
</tbody>
</table>

Table 2 Comparison of RMS values of off-diagonal terms in cross-orthogonality check using Static, Hybrid and Present TAMs.
4 Conclusions

The orthogonality checks as obtained using the present TAM with residual space modes are as good as the orthogonality using the hybrid TAM. About half the off-diagonal terms in the cross-orthogonality matrix are lower than the corresponding terms via the hybrid method and about half higher. This pattern seems to be consistent when a number of different structures were analyzed for the orthogonality performance using the present and hybrid TAMs. It is noted that the number of modes that can be used in a TAM generation is limited by the number of 'a'-set DOF. This leads to the conjecture that TAMs that attempt to retain the residual flexibility together with the target mode representation, albeit as an approximation, are superior to TAMs that have only the target mode representation, and the difference in the performance of such TAMs is marginal.

Acknowledgments

The authors wish to thank the University Research Council of the University of Cincinnati for supporting the present work.

REFERENCES