ABSTRACT

In this work, algorithmic approaches to enhance system health monitoring capability are developed in which direct frequency response function data is utilized. The use of direct frequency response functions, as opposed to measured model parameters, are shown to be one method to address a part of the incomplete measurement problem common to model correlation and system health monitoring; namely the mismatch in the number of measured vibration modes in the measured frequency band in comparison to the number of modes included in the analytical finiteelement model. In addition, there are several advantages to using direct frequency response function measurements, including: (i) reduction in testing time, in that typically only a few number of frequency values and (ii) reduction in analysis effort, in that the skill of modal parameter curve fitting is eliminated. Of course, it is well recognized that the advantage of reducing measurement noise via modal curve fitting must be addressed by choosing “regions” of the frequency response function which are not adversely corrupted by noise or using other numerical techniques to reduce the effects of noise.

1. INTRODUCTION

Structural health monitoring/assessment of complex systems such as aircraft, off-shore platforms, bridges and buildings is required to maintain adequate levels of safety and performance. The most common method for current structural damage detection programs is through visual inspection. This method is costly and in many cases difficult to perform due to the inaccessibility of major portions of the structure. In addition, visual inspection does not give a quantitative value for the remaining strength of the structure. More recently, non-destructive testing procedures have been employed to assess the integrity of structures. These methods include radiographic, X-ray, acoustic emission, ultrasonic and system identification techniques. Of particular interest is system identification techniques, which utilize changes in the vibration signature of the structure and an original analytical model of the undamaged structure to determine both the location and extent of structural damage.

Algorithms used to address the structural health monitoring (as well as model correlation) can be broadly classified as falling into one of four different approaches—optimal—matrix updates, sensitivity methods, eigenstructure assignment techniques, and minimum-rank perturbation methods. Surveypapers providing an overview of some of these techniques are provided in Refs. [1,2].

In the optimal matrix update formulation, perturbation matrices for the mass, stiffness, and/or damping matrices (property matrices) are determined which minimize a given cost function subject to various constraints. A typical cost function used is the Frobenius norm of the perturbation matrix [3]. Typical constraints may include satisfaction of the eigenproblem for all measured modes, definiteness of the updated property matrices and preservation of the original sparsity pattern.

Sensitivity methods for damage detection and model refinement make use of sensitivity derivatives of modal parameters with respect to physical design variables [4] or with respect to matrix element variables [5]. When varying physical parameters, the models remain consistent within the original Finite Element (FE) program framework.

Control-based eigenstructure assignment techniques determine the pseudo control which would be required to produce the measured modal properties with the initial structural model [6,7]. The pseudo control is then translated into matrix adjustments applied to the initial FEM.

Finally, the development of a minimum rank perturbation theory has been recently proposed as a computationally attractive approach for system health monitoring and model correlation [8]. The update to each property matrix is of minimum rank, and is equal to the number of experimentally measured modes which the modified model is to match.

The vast majority of proposed system identification techniques utilize measured modal data (natural frequencies, damping ratios, and mode shapes) as direct “inputs”. In other words, the vibration signature of the structure is characterized by its modal parameters. The modal parameters are typically obtained from measured frequency response functions and/or direct time-domain identification algorithms.

The use of direct frequency response functions in model correlation and damage detection studies has been investigated recently. The development of an “exact” technique to locate (but not determine extent) modeling errors via the development of a frequency response function residual has been investigated [9]. The effect of various objective function formulations utilizing frequency response function information was investigated in [10]. In this study, the problem was solved using a gradient descent numerical optimization pack-
age. In [11], a large overdetermined set of equations is derived based on the generalized matrix inversion equation specifically applied to a mixed analytical/experimental frequency response function equation. These equations are then solved for either (i) individual elements of the stiffness matrix or (ii) physical parameters using a sometimes iterative least squares solution. A sensitivity analysis of the FRF with application to model correlation is presented in [12]. This technique is used with experimental data to update a model of a plate and of a framed structure. Damping is ignored in one case and estimated from experimental data in another. Another sensitivity analysis is presented in [13] with model correlation algorithm development being performed in [14].

In this work, we propose a system identification procedure which directly utilizes measured frequency response functions. There are several advantages to using direct frequency response function measurements, including: (i) reduction in testing time, in that typically data is required at only a few number of frequency values and (ii) reduction in analysis effort, in that the skill/art of modal parameter curve fitting is eliminated. Of course, it is well recognized that the advantage of reducing measurement noise via modal curve fitting must be addressed by choosing "regions" of the frequency response function which are not adversely corrupted by noise.  

The proposed approach the frequency response function measurements, in conjunction with minimum rank perturbation theory [3], is used to arrive at perturbations to the stiffness matrix. Inspection of the perturbation matrix provides information concerning both the location and extent of structural damage.

2. MINIMUM RANK PERTURBATION THEORY

2.1 Theoretical Background - MRPT

In this section, the theoretical foundation of the Minimum Rank Perturbation Theory (MRPT) is presented.

**PROPOSITION 1** - Suppose that $X, Y \in \mathbb{R}^{m \times p}$ are given where $p < m$ and $\text{rank}(X) = \text{rank}(Y) = p$. Define $\mathcal{C}$ to be the set of matrices $A$ in $\mathbb{R}^{m \times m}$ that satisfies the problem,

$$AX = Y \quad \text{with} \quad A^T = A \quad (1)$$

The inverse of $\mathcal{C}$ is nonempty, the minimum rank of any matrix $A$, in $\mathcal{C}$ is $p$.

Next, define $\mathcal{C}_p$ to be a subset of $\mathcal{C}$ comprised of all $A$ such that $\text{rank}(A) = p$.

Then

1. (a) If the matrix $\mathbf{Y}^T X$ is symmetric, then one member of $\mathcal{C}_p$ is given by,$$
\begin{align*}
A^p &= \mathbf{Y}^T \mathbf{H} \mathbf{Y}^T \quad \text{with} \quad \mathbf{H} = (\mathbf{Y}^T X)^{-1} 
\end{align*}
$$

2. (b) The matrix defined by Eq. (2) is the unique member of $\mathcal{C}_p$.

The proof of Proposition 1 is treated in Ref. [8]. The proposition I proposition 1, matrix $H$ in Eq. (2) is invertible because matrix $Y$ is assumed to be of full rank. The next proposition addresses the case when matrix $Y$ is rank deficient.

**PROPOSITION 2** - Suppose that $X, Y \in \mathbb{R}^{m \times m}$ are given with $\text{rank}(X) = m$ and $\text{rank}(Y) = p$, where $p < m < n$. Further, suppose that the matrix $\mathbf{Y}^T \mathbf{X}$ is symmetric.

Define $\mathcal{U}$ to be the set of matrices $A$ in $\mathbb{R}^{m \times m}$ that satisfies the problem,

$$A^p X^p = Y^p \quad \text{with} \quad A^T = A^p \quad (3)$$

where the superscript $p$ indicates a rank $p$ matrix. In Eq. (3), $X^p, Y^p \in \mathbb{R}^{m \times p}$ are corresponding full rank submatrices of $X$ and $Y$.

Then

The set $\mathcal{U}$ contains a single member, $A_p$, that can be calculated from Eq. (2) using any corresponding $X^p$ and $Y^p$. The detailed proof of the above proposition can be found in Ref. [8].

2.2 Stiffness Modification Using Frequency Response Function Data

The development of the frequency response MRPT approach is based on the frequency-domain equations of motion

$$Z(\omega)X(\omega) = F(\omega) \quad (4)$$

where $X(\omega)$ is the steady-state displacement profile, $F(\omega)$ is the steady-state applied force vector, and $Z(\omega)$ is the impedance matrix given by

$$Z(\omega) = -\omega^2M + i\omega C + K \quad (5)$$

In Eq. (5), $M$, $C$, and $K$ are the analytical mass, damping and stiffness matrices respectively, $i=\sqrt{-1}$, and $p$ is the number of structural DOFs. In general, the "true" frequency-domain equations of motion can be written as

$$Z(\omega) + \Delta Z X(\omega) = F(\omega) \quad (6)$$

where $\Delta Z$ represents the effect of damage on the impedance matrix (assuming that damage only affects the structural stiffness matrix, $\Delta Z = \Delta K$). Assuming that the frequency response function has been measured at $q$ distinct frequencies, Eq. (6) can be written for all measurements as

$$\Delta Z = Z(\omega)X(\omega_1)X(\omega_2) ... X(\omega_p) = \begin{bmatrix} F(\omega_1) - Z(\omega_1)X(\omega_1) & \cdots & F(\omega_p) - Z(\omega_p)X(\omega_p) \end{bmatrix} \quad (7a)$$

or

$$\Delta Z = Y(\mathbf{Y}^T X)^{-1} Y^T \quad (7b)$$

Note that the right hand side of Eq. (7b) is essentially a frequency response function residual. Damage degrees of freedom are indicated by nonzero rows of $Y$ in Eq. (7b).

Letting $A = \Delta Z$ and comparing Eq. (7b) to Eq. (1), the minimum rank perturbation solution to Eq. (7b) is given by

$$A = Y(\mathbf{Y}^T X)^{-1} Y^T \quad (8)$$

Note that the inverse required is of a matrix whose dimension is $p \times p$, where $p$ is the number of distinct frequencies that are used in the update (equal to or less than the number of frequency points measured). Assuming that damage has occurred at only a single truss element, such that damage is reflected in the stiffness matrix as a rank one perturbation, Eq. (6) shows that $X(\omega)$ and $E(\omega)$ need only be measured at one frequency. This demonstrates the great potential for reducing.
both experimental and analysis time for real-time damage assessment.

2.3 Selection of Frequency Response Function Data

Frequency response functions are typically measured at a power of 2 number of discrete frequencies, with typical spectrums consisting of 512 to 4096 discrete frequency points. Thus, a practical concern is how to select both the number and specific frequencies which will be used in the MRPT update. The first step in “data” reduction is to eliminate those “regions” of the frequency response function which exhibit low coherence due to either noise, non-linearities, and/or unmeasured excitation at that specific frequency. With these “regions” eliminated, a previously developed subspace selection algorithm can be used to extract the “most-consistent” information out of the remaining data. A brief review of the subspace selection algorithm follows.

For all types of data, the MRPT perturbation matrix constraint equation is given in generalized form by Eq. (1). The subspace selection algorithm is defined to be the search for a matrix \( Q \in \mathbb{R}^{p \times f} \) such that

\[
AXQ = YQ \tag{9}
\]

is well conditioned. The unknowns at this point are \( \hat{p} \), the numerical rank of \( Y \) and \( Q \). Consider the singular value decomposition of \( Y \) defined as

\[
Y = [U_1, U_2, \ldots, U_n] \begin{bmatrix}
\Sigma & 0 \\
0 & 0
d \end{bmatrix} \begin{bmatrix}
V_1 & V_2 \\
V_3 & V_4
\end{bmatrix} \tag{10}
\]

where \( \Sigma \in \mathbb{R}^{p \times p} \) is the matrix of non-zero singular values (or singular values greater than some prescribed tolerance) and the left and right singular vectors \( U \) and \( V \) are partitioned conformable. When \( Y \) is rank deficient, the range of \( Y \) is spanned by the \( \hat{p} \) columns of \( U_1 \). Thus, it is desired to find a CI such that

\[
YQ = U_1 \tag{11}
\]

The matrix \( Q \) can be approximated by utilizing the pseudo inverse of \( Y \) as

\[
Q = Y^+U_1 = V_1\Sigma^+U_1 = V_1\Sigma^+ \tag{12}
\]

Note that the left singular vector \( U_1 \) does not have to be explicitly formed and saved. The calculation of \( Q \) using the Chan SVD \( [15] \) is \( n_p^2 + \frac{17}{3} n_f^3 \) flops. Note importantly that the flop count varies linearly with FEM dimension. The solution to Eq. (9) is then given by

\[
A = YQ(Q^TYTXQ)^{-1}Q^TY^T \tag{13}
\]

3.0 Examples

3.1 Test-Bed Structural Description

The structure investigated is the eight-bay cantilevered two-dimensional truss shown in Figure 1. This structure was designed \([16]\) to emulate typical properties of space structures: low frequency modes with large non-structural masses. The geometric and material properties of the truss are given in Figure 1. The truss consists of 40 struts and 16 nodes.

Fourteen of the nodes are loaded with concentrated non-structural masses of magnitude \( 1 \text{ lbs-sec}^2/\text{in} \) and the remaining two nodes have large lumped masses of magnitude \( 10 \text{ lbs-sec}^2/\text{in} \). Each truss strut was modeled as a rod element with a density of \( 9.0 \times 10^{-3} \text{ lbs-sec}^2/\text{in}^4 \). The finite element model has 32 translational DOFs (2 DOFs per node). The truss, as described above, is considered the healthy (undamaged) configuration in this study. The driving point for the FRF was chosen as depicted in Figure 1.

![Figure 1 - The Eight-Bay Two-Dimensional Truss](image)

\[
EA = 1.525 \times 10^7 \text{ psi} \quad m = 1 \text{ lbs-sec}^2/\text{in} \quad M_1 = M_2 = 10 \text{ lbs-sec}^2/\text{in} \quad g = 9.0 \times 10^{-3} \text{ lbs-sec}^2/\text{in}^4
\]

3.2 Frequency Selection Techniques

Many times the number of frequency points that can be used is limited either by the testing conditions or by available computational power. Sometimes only a small number of points may be measured due to the large time per frequency point cost, as with a sine dwell test. Conversely, if a large number of frequency points may be generated, such as a broadband random test, is used the maximum size \( Y \) that an SVD can be performed on may be limited by the available computers. To determine how to choose which frequency points to use in the analysis, six different frequency selection techniques are used in conjunction with four different damage cases. In this study, 512 equally spaced frequency points were first generated over a range of \( 0-100 \text{ rad/sec} \). Then five subsets of the original data were chosen. Figure 2 shows pictorially the six different frequency techniques with stars denoting the chosen points. The FRFs shown have been multiplied by a gain. The number of frequency points used is dependent on the number of “clustered” regions. The six choices of frequency points are:

1. All 512 frequency points.
2. Selection of 52 evenly space frequency points.
3. Selection of 51 frequency points “clustered” around resonances.
4. Selection of 42 frequency points “clustered” around anti-resonances of the FRF in the Y-direction of the driving point.
5. Selection of 44 frequency points distributed away from resonances and anti-resonances.
6. Selection of 70 frequency points selected from percent change difference of healthy/damaged frequency response functions. Figure 3 shows the selected points and the log10(error) where error = \( X_d(\omega_j) \cdot X_h(\omega_j) \), with \( X_d(\omega_j) \) and \( X_h(\omega_j) \) representing the damaged and healthy FRFs respectively, and \( j \) represents an element by element division. The largest number of points were selected in the region of largest change. Than points were chosen around the small peaks to provide some distribution.
Figure 3. FRF Case 6 and Change in FRF

Figure 3. FRF Case 6 and Change in FRF

Figure 4 depicts the four damage cases investigated. The darkened elements indicate the damaged struts. The four damage cases are formed by starting at a low level of damage where three struts are damaged and progressively increasing the number of damaged struts until a maximum of 14 damaged struts are reached. Each new case of damage has the previous case as a subset. The damage is modelled as a 10% decrease in stiffness and the struts to be damaged were randomly chosen. In all of the damage cases there are a mixture of low and high strain members.

Figure 3. Damage Cases

The number of damaged members for each case was guided by the number of modes in the bandwidth of study (0–100 rad/sec). Previous modal based implementations of MRPT required the same number of modes as the rank of the damage to calculate the exact extent. For the rod element used here, this would be equal to the number of damaged elements. As seen in Fig. 2, the structure has six modes within the measurement bandwidth, therefore, MRPT based on modal parameters would be able to correctly identify up to a six element damaged truss (damage cases 1 and 2). Damage cases 3 and 4 are used to explore how much additional information is available for a FRF based MRPT extent calculation.

To summarize, the example consists of six different methods of selecting frequency points from a data set of 512 points. Each of these methods are applied to each of the four cases of damage. Once the frequency points and the damage case is chosen, Eq. (7) is used to calculate the residual (Y). An SVD is then performed on Y as detailed in Section 2.3 and the rank of Y is determined. The AZ is then calculated (Eq. (13)) and compared with the actual perturbation matrix.

The results of the study are summarized in Table 1. The error in the extent calculation is characterized by

$$\text{Error} = \frac{|\Delta K_d - \Delta K_{cal}|}{|\Delta K_d|}$$  

where $\Delta K_d$ is the theoretical perturbation which reflects damage and $\Delta K_{cal}$ is the perturbation computed using the MRPT technique presented above.
Table 1. Error Summary of Case studies

<table>
<thead>
<tr>
<th>FRF Case</th>
<th>Number of FRF Points</th>
<th>Damage Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Error Case 1</td>
</tr>
<tr>
<td>1</td>
<td>512</td>
<td>2e-11</td>
</tr>
<tr>
<td>2</td>
<td>52</td>
<td>4e-11</td>
</tr>
<tr>
<td>3</td>
<td>51</td>
<td>3e-10</td>
</tr>
<tr>
<td>4</td>
<td>42</td>
<td>3e-12</td>
</tr>
<tr>
<td>5</td>
<td>44</td>
<td>7e-12</td>
</tr>
<tr>
<td>6</td>
<td>70</td>
<td>2e-10</td>
</tr>
</tbody>
</table>

† Calculated using a rank 13 update.

Frequency response case 1 should show the minimum error since all the points are being used. In damage case 1 and 2, all the errors are within the round off error of the computer. In damage cases 3 and 4 the minimum error occurs in FRF case 1, as expected.

The two lowest levels of damage (three and six members damaged) are accurately estimated for all choices of FRF points. An accurate extent calculation for damage case 1 and 2 would be expected since there are a sufficient number of modes available to predict the damage.

The two highest levels of damage (12 and 14 members damaged) are poorly estimated when frequency points are chosen around the resonances and around the anti-resonances. For the highest level of damage (14 elements damaged), there was insufficient information as determined from the singular values of Y to do a full 14 rank update. A reduced update was performed (rank 13). Damage extent was accurately calculated for all damage cases when points were chosen away from the resonances and anti-resonance (FRF case 5). These observations are consistent with those made by Lammens who used a sensitivity based algorithm with the FRF to calculate the perturbation matrices [12].

Damage case 4 has a high error for FRF case 6 (points chosen around large changes in the FRF). The selected points are clustered around the resonances and anti-resonances, hence, the results are similar to the results of FRF case 3 and 4 where the points are clustered around the resonances and anti-resonances, respectfully.

3.3 Evenly Spaced Point Selection

In this study, it appears that an evenly spaced frequency point selection is a reasonable technique for data reduction. To investigate this further, two additional studies were performed on damage case 3. In the first study, the frequency bins selected were shifted from the original selection by an integer ranging from 0-8. The shifting did not affect the spacing of the points, only the points which were used. The second study consisted of increasing the spacing of the frequency points, thereby decreasing the number of points used.

The shifting of the frequency bins has little affect on the overall error (Figure 4). The error remains below 1e-5 with roundoff error causing the variation. Figure 5 graphically shows the results of increasing the spacing of the frequency points. The error remains relatively constant up to the point that 14 frequency points are used. With less than 14 points, the error begins to rapidly increase. The perturbation matrix being calculated is rank 12. Thus, it would be anticipated that as the number of frequency points approaches 12, the error would rise. The fact that the error is below 10^-3 when using 12 or more frequency points indicates in this study that the evenly spaced data criteria is a strong candidate for guiding frequency point data selection.

**CONCLUSIONS**

An implementation of MRPT using frequency response data is presented. The use of FRFs directly simplifies the analysis by avoiding the extraction procedure of the modal parameters. The use of FRF data also allows a larger rank update than the number of modes in the bandwidth would seem to allow.

Issues associated with the choice of frequency points is also explored. It is found that regions away from resonances or anti-resonance contain the richest data. Concentrating points at resonances or anti-resonances or at areas where large changes in the FRF occur (due to damage), while intuitively appealing, does not provide the "best" results in this particular study.

**REFERENCES**


