Parameter Identification for Nonlinear Hysteresis Damping with Spectral Analysis Method

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Abstract: The hysteresis damping characterization of materials and structures is difficult to determine due to their nonlinearity at strain levels of interest to engineers. A method is presented for the determination of damping characteristics of the materials with steady-state hysteresis loops for a harmonic foundation motion. The nonlinear hysteresis properties of the tested materials are identified with the aid of the spectral analysis method. The nonlinearity of the material creates the multiple frequencies in the response force signal. The associated parameter of multiple frequencies can be identified with this nonlinear system parameter identification technique. A program in MATLAB was developed to identify the parameters of the first three multiple frequencies with the aforementioned method. In principle, the method offers the possibility of identifying the parameters of higher multiple frequency, which usually have a negligible effect for mechanical systems. The higher order nonlinear terms of rubber components were identified successfully by this newly developed method.

NOMENCLATURE

- \( E_a \): tangent modulus (or spring constant) on initial load
- \( F(o) \): amplitude spectrum of a force function
- \( H \): a geometrical parameters of hysteretic loops
- \( H(w) \): transfer function
- \( J \): damping constant
- \( P \): force
- \( P_k \): Fourier coefficient of a force function
- \( P_x \): force amplitude in a hysteretic loop
- \( x \): displacement
- \( X_d \): displacement amplitude in a hysteretic loop
- \( u \): energy loss per cycle
- \( f \): frequency or loading frequency
- \( k \): rank of order
- \( m \): the order of highest multiple frequency
- \( n \): damping exponent
- \( p_k \): coefficient of a force function
- \( t \): time
- \( \phi \): phase or phase delay
- \( \phi(\omega) \): phase spectrum of a force function
- \( \sigma \): stress
- \( \sigma_x \): stress amplitude in a hysteretic loop
- \( \varepsilon \): strain
- \( \omega \): angular frequency
1. INTRODUCTION

In order to utilize damping in the analysis of vibration and other types of dynamic systems, the damping properties may be expressed either in terms of (a) a damping force (such as dashpot or viscous force), (b) the energy loss per cycle (the area within the hysteresis loop), or (c) the P-X (or σ-ε) equations for the hysteretic loop.

Among the above three representations, one of the most widely used is the damping force representation, which assumes that the damping force is proportional to the velocity, and is generally represented schematically by a dashpot. It is appropriate for linear and nonlinear rate-dependent damping. The disadvantage of the damping force is that it is a contrived model, which only very rarely represents what happens in the real world. Engineers find that the most actual mechanical system do not follow the prediction using simple viscous damper. Hence, it is not generally suitable for a rate-independent damping dissipation that frequently leads to nonelliptical hysteretic loops.

The energy loss per cycle of vibration is one of the most common methods to determine the magnitude of structural damping. Materials and structures are often tested by measuring force (stress) and displacement (strain) under carefully controlled steady-state harmonic loading. Such test produce hysteretic loops. Although damping-stress relations are very complex, for some mechanisms this relation can be expressed as follows:

\[ U = J P^n \]  

There are two subclasses of damping: (a) linear damping or quadratic damping (damping exponent \( n=2 \)), of which the hysteretic loop is elliptical in form; (b) nonlinear damping or nonquadratic damping (damping exponent \( n \) has value other than 2, usually greater than 2), of which the hysteretic loop is not elliptical in form. Several types of linear and nonlinear damping loops[1] are shown in Figure 1. The damping exponent \( n \) in Equation (1) is a function of stress amplitude. The damping exponent is introduced in most texts to describe the nonlinearity of hysteretic loops. Nevertheless, the relationship between damping exponents and the shapes of relative loops have not been clearly described by previous investigators. Besides, the contribution of the effect of nonlinearity to energy loss per cycle of nonlinear hysteretic loops has never been proposed. In the study of the above mentioned unsolved problem, the authors have developed a new P-X equation, in which the multiple frequencies from the effect of nonlinearity of tested specimens have been used to represent the shape of any hysteretic loop[2]. This new representation has proved to be applicable for almost any kind of the hysteretic loops. This paper mainly describes the parameter identification technique for the nonlinear hysteretic damping.

![Figure 1: Typical linear and nonlinear hysteretic loops[1].](image)

2. NONLINEAR HYSTERETIC LOOPS AND DAVIDENKOV EQUATION

The linear mechanism can be characterized by two features: quadratic damping and an elliptical hysteretic loop. As the stress amplitude increases, particularly near the fatigue limit, both the
magnitude of damping and the degree of nonlinearity generally increase. Thus, the corresponding hysteretic loops of material will no longer be elliptical. Besides, the structural damping from friction between the interfaces of parts will also create a nonelliptical loop. Loops (c) to (f) in Figure 1 are typical nonlinear hysteretic loops. All the loops can be divided by two branches, loading branch and unloading branch. Usually branches curve intersections may produce shape corners (i.e. singularities). However, the branch intersections may be somewhat rounded for materials under some conditions because the shape of hysteretic loop change gradually from elliptical loop to nonelliptical loop as the effect of nonlinearity increase with the introduction of singularities.

One famous expression for the P-X loop is the Davidenkov equations. In the Davidenkov approach, each branch of the loop has a different functional form:

\[ \sigma = E \left[ \varepsilon - \frac{H}{u} \left( \varepsilon_a + \varepsilon - 2^{n-1} \varepsilon_a \right) \right] \]
\[ \sigma = E \left[ \varepsilon - \frac{H}{u} \left( \varepsilon_a - \varepsilon - 2^{n-1} \varepsilon_a \right) \right] \]  

(2)

The Davidenkov expressions have been used extensively to solve several classes of nonlinear problems in which the dissipation elements are assumed to have hysteretic loops of the form shown in loop (d) of Figure 1. Notice that this kind of loop always has a cusped shape with the two corners inclined to the horizontal. The loop (d) in Figure 2 is only one kind of a number of different shapes of hysteretic loops. Thus, there is a need for a P-X equation to represent any arbitrary shape of hysteretic loops.

3. A NEW P-X EQUATION FOR NONLINEAR HYSTERETIC LOOPS

Most commonly, specimens are tested under carefully controlled steady-steady sinusoidal loading, either by a sinusoidal force load or sinusoidal displacement load. For this work, it is assumed that the measurement has a sinusoidal displacement loading:

\[ x = X \cos \omega t \]

(3)

The higher order terms of nonlinearity of the materials will reasonably create multiple frequencies in the corresponding force function. The effect of this multiple frequencies will change the hysteretic loop from an elliptical loop to a nonelliptical one.

A new P-X equation presented in this paper can be written as a simple parametric equation:

\[ X = X_a \cos \omega t \]
\[ P = \sum_{k=1}^{m} P_k \cos(\phi_k + \phi) \]  

(4)

The above P-X parametric equation can also be represented in a closed form with exponential notations:

\[ P = \sum_{k=1}^{m} P_k X_a e^{i(k \omega t + \phi)} \]  

(5)

The new P-X equation has the following improvements over the Davidenkov Equation:

- It can represent any arbitrary steady-steady hysteretic loop in a closed form equation with dramatically increased accuracy. Usually the first three terms of the Equation can match most practical nonlinear loops well enough. Whereas, the shape of applicable loops for Davidenkov is very limited.

- The tangent modulus no longer play a role in the new P-X equation, thus it removes the extra work of measuring the tangent modulus during the initial load. The effect of nonlinearity in the specimens looks more reasonable when the multiple frequency effect of nonlinearity has been introduced.
The new equation can be used to produce locus curves for a series of nonlinear hysteretic loops at different displacement (strain) amplitude as long as one loop is known. The parameters $\beta_k$ of Equation (5) is a function of displacement amplitude and can be worked out by including the effect of higher order terms of the nonlinearity within the concentrated displacement (strain) range. Whereas, the Davidenkov equation has no such ability because the parameter $u$ and $H$ usually are not constant for a given material, particularly at larger amplitude of stress.

The new equation offers a bridge to study the contribution of the effect of nonlinearity to energy loss per cycle of hysteretic loops. The new equation has helped the authors to describe the relationship between damping exponents and the shapes of force-displacement loops, which has not been mentioned by previous investigators. The value of energy loss per cycle is dependent on the fundamental frequency term only (see Equation (4)). The effects of multiple Frequency terms change the shape of the loop but do not change the energy loss per cycle.

The equation to calculate damping exponent:

$$n = 2 \frac{\log P_1}{\log P_a} \quad (6)$$

Most nonlinear hysteretic loops are associated with plastic strain effects. The corners of the loops are usually inclined to the horizontal. Thus, most often, the damping exponent has a value between 2 and 3 for the intermediate stress range. Beyond a certain critical stress, however, the effect of nonlinearity becomes very strong. Thus, the damping exponent may be as high as 20.

4. PARAMETER IDENTIFICATION WITH SPECTRUM ANALYSIS METHOD

One of the main tasks for characterization of the nonlinear hysteretic damping is to determine the parameters in Equation (4) for a given hysteretic loop. These parameters include $m$, $p_k$, and $\phi_k$. A parameter identification method associated with the spectral analysis has been developed by the authors. This method has been used successfully to identified the nonlinear parameters in different hysteretic loops.

During the discrete signal processing in the measurement and parameter identification, the sinusoidal displacement and corresponding force signal are needed to identify the parameters. In the case that only the steady-state loop is known, one may first divide the whole loop to several reasonable pieces and uses piecewise polynomials to represent these separated curves. Then, constructs a set of discrete sinusoidal displacement data for one cycle and calculate the corresponding discrete force data from these piecewise polynomials. Finally, these two sets of data are extended to a number of cycles to satisfy the requirement of FFT analysis.

The value of $m$ is corresponding to the order of the highest multiple frequency needed for Equation (4). Generally, Equation (4) can match most loops well with only three terms. In other words, the higher order terms above third terms usually have a negligible contribution, for engineering purposes. In this paper, all equations for parameter identification method show the first three terms.

One may extend this method to identify any higher order terms without difficulty.

Figure 2 shows the model for the parameter identification method used in this investigation[3]. Three distinct transfer functions can be found from separating $X$ into a linear and two nonlinear terms in the spectral analysis. The signal is separated into $X(t)$, $X^3(t)$, $X^3(t)$ and $P(t)$.

The transfer function value at $\omega$ between $X(t)$ and $P(t)$ is:

$$H_1(\omega) = |H_1(\omega)|e^{i\phi_1(\omega)} \quad (7)$$

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The transfer function value at $2\omega$ between $X^2(t)$ and $P(t)$ is:

$$H_2(2\omega) = |H_2(2\omega)|e^{i\phi_2(2\omega)} \quad (8)$$

The transfer function value at $3\omega$ between $X^3(t)$ and $P(t)$ is:

$$H_3(3\omega) = |H_3(3\omega)|e^{i\phi_3(3\omega)} \quad (9)$$

One can obtain the following relationships by combining powers of trigonometric functions with Equation (4) and Equation (5):

$$P_1 = |H_1(\omega)| \quad \phi_1 = \phi_1(\omega)$$

$$P_2 = \frac{1}{2}|H_2(2\omega)| \quad \phi_2 = \phi_2(2\omega)$$

$$P_3 = \frac{1}{4}|H_3(3\omega)| \quad \phi_3 = \phi_3(3\omega) \quad (10)$$

Submitting above equation to Equation (5) gives

$$P = |H_1(\omega)|X_a e^{i(\omega t + \phi_1(\omega))}$$

$$+ \frac{1}{2}|H_2(2\omega)|X_a^2 e^{i(2\omega t + \phi_2(2\omega))}$$

$$+ \frac{1}{4}|H_3(3\omega)|X_a^3 e^{i(3\omega t + \phi_3(3\omega))} + \ldots \quad (11)$$

With the above method, first three parameters of any steady-state hysteretic loop with a given displacement amplitude can be identified.

### 5. EXPERIMENTAL STUDY ON MATERIAL DAMPING CHARACTERISTIC OF RUBBER

A MATLAB program based on above identification method have been used to identify the relative parameters of hysteretic loops. The developed parameter identification method was used for damping parameter measurement of a pair of rubber couplings. A loop was obtained during the measurement as shown in Figure 3(a). The hysteretic loop is much like an ellipse, which indicates the relationship of strain and stress is quite linear at this stress level. By applying the new parameter identification method, the authors found nonlinear terms in the rubbers, even though the effect of the nonlinearity was very small as can been seen in Figure 3(d) and Figure 3(e).

The P-X representation of the loop of rubber couplings can be worked out respectively:

$$P \approx 33.3 X_a e^{i(\omega t + 0.23)}$$

$$+ 1.18 X_a^2 e^{i(2\omega t - 1.19)}$$

$$+ 78.8 X_a^3 e^{i(3\omega t - 1.03)}$$

where $X=0.1$ in. and $\omega=2\pi f$ ($f=20$ Hz).

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Figure 3: An example of nonlinear parameter identification for rubber mount.
6. CONCLUSIONS

The presented investigation of hysteretic damping and application of the new method to hysteretic loops suggest the following conclusions:

- The effect of nonlinearity in the hysteretic material and structures will cause the multiple frequencies in the corresponding force function if a specimen has a sinusoidal displacement loading.

- A new polynomial type P-X equation which includes the effect of multiple frequency component has been developed. It shows an improvement over the Davidenkov equation and can represent any arbitrary steady-steady hysteretic loop in a closed form equation with increased accuracy.

- The energy loss per cycle and the damping exponent of hysteretic loops have been clearly interpreted with the new P-X equation. The value of energy loss per cycle is dependent on the first term of P-X equation only. The effects of multiple frequency terms change the shape of the loop but do not change the energy loss per cycle. A whole loop is described as the contribution from each multiple frequency component, which has its own loop pattern with a certain phase delay.

- A parameter identification technique in conjunction with spectral analysis method has been used to identify the loop parameters. It has been proved to be accurate and repeatable.

- The small nonlinear terms of rubber specimens have been identified successfully with the new nonlinear parameter identification method.

REFERENCE:


