Cyclic Averaging for Frequency Response Function Estimation

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ABSTRACT

Cyclic averaging, as a method for reducing the leakage error in the estimation of frequency response functions, is a simple and effective method that can be implemented in any data acquisition system. With the widespread availability of multi-channel data acquisition systems, cyclic averaging may be the easiest approach to minimizing the truncation effect, known as leakage, that occurs when time domain data is transformed to the frequency domain utilizing fast Fourier transforms. Cyclic averaging, historically, has been used in rotating equipment data analysis to enhance characteristics in the data that are functions of the rotating speed. This same data averaging technique can be used when estimating frequency response functions to enhance the frequency characteristics in the data that are integer functions of the observation window. In this way, the nonperiodic characteristics in the data are minimized and leakage is reduced. A presentation of the theory of cyclic averaging and its application and effectiveness in a typical frequency response function estimation procedure is included.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>Δf</td>
<td>Frequency resolution</td>
</tr>
<tr>
<td>C_{pk}</td>
<td>Complex Fourier coefficients for ω_k</td>
</tr>
<tr>
<td>f(t)</td>
<td>Force, time domain</td>
</tr>
<tr>
<td>F(ω)</td>
<td>Force, frequency domain</td>
</tr>
<tr>
<td>FRF</td>
<td>Frequency Response Function</td>
</tr>
<tr>
<td>n, k, N</td>
<td>Integers</td>
</tr>
<tr>
<td>H_1, H_2, H_n</td>
<td>FRF Estimation, noise on output</td>
</tr>
<tr>
<td>N_a, N_c</td>
<td>Number of asynchronous averages</td>
</tr>
<tr>
<td>t_o, T</td>
<td>Time offset, time period of observation</td>
</tr>
<tr>
<td>x(t)</td>
<td>Response, time domain</td>
</tr>
<tr>
<td>x_i(t)</td>
<td>Response, i-th average, time domain</td>
</tr>
<tr>
<td>X(ω)</td>
<td>Response, frequency domain</td>
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1. Introduction

The averaging of signals is normally viewed as a summation or weighted summation process where each sample function has a common abscissa. Normally, the designation of history is given to sample functions with the abscissa of absolute time and the designation of spectra is given to sample functions with the abscissa of absolute frequency. The spectra are normally generated by Fourier transforming the corresponding history. In order to generalize and consolidate the concept of signal averaging as much as possible, the case of relative time can also be considered. In this way relative history can be discussed with units of the appropriate event rather than seconds and a relative spectrum will be the corresponding Fourier transform with units of cycles per event. This concept of signal averaging is used widely in structural signature analysis where the event is a revolution. This kind of approach simplifies the application of many other concepts of signal relationships such as Shannon’s sampling theorem and Rayleigh’s criterion of frequency resolution.

2. Background

The process of signal averaging, as it applies to frequency response functions, is simplified greatly by the intrinsic uniqueness of the frequency response function. Since the frequency response function can be expressed in terms of system properties of mass, stiffness, and damping, it is reasonable to conclude that in most realistic structures, the frequency response functions are considered to be constants just like mass, stiffness, and damping. This concept means that when formulating the frequency response function using $H_1$, $H_2$, or $H_n$ algorithms, the estimate of frequency response is intrinsically unique, as long as the system is linear and the noise can be minimized or eliminated. In general, the auto- and cross-power spectra are statistically unique only if the input is stationary and sufficient averages have been taken. Generally, this is never the case. Nevertheless, the estimate of frequency response is valid whether the input is stationary, non-stationary, or deterministic.

The concept of the intrinsic uniqueness of the frequency response function also permits a greater freedom in the testing procedure. Each function can be derived as a result of a separate test or as the
In either case, the estimate of frequency response function will be the same as long as the time history data is acquired simultaneously for the auto- and cross-power spectrums that are utilized in any computation for frequency response or coherence function.

Generally, averaging is utilized primarily as a method to reduce the error in the estimate of the frequency response function(s). This error can be broadly considered as noise on either the input and/or the output and can be considered to sum of random and bias components. Random errors can be effectively minimized through the common approach to averaging. Bias errors, generally, cannot be effectively minimized through the common approach to averaging. Cyclic averaging, however, does minimize the bias error caused by the truncation of the time domain signals in conjunction with the use of a discrete Fourier transform (DFT). This error is commonly known as the leakage error.

The approaches to signal averaging for these different situations vary only in the relationship between each sample function used. Since the Fourier transform is a linear function, there is no theoretical difference between the use of histories or spectra. (Practically, though, there are precision considerations.) With this in mind, the signal averaging useful to frequency response function measurements can be divided into three classifications:

- Asynchronous
- Synchronous
- Cyclic

These three classifications refer to the trigger and sampling relationships between sample functions.

### 2.1 Asynchronous Signal Averaging

The classification of asynchronous signal averaging refers to the case where no known relationship exists between individual sample functions. The FRF is estimated solely on the basis of the intrinsic uniqueness of the frequency response function. In this case, the power spectra (least squares) approach to the estimate of frequency response must be used since no other pathway of preserving phase and improving the estimate is available. In this situation, the trigger to initiate digitization (sampling and quantization) takes place in a random fashion dependent upon the equipment availability. The triggering is said to be in a free-run mode.

### 2.2 Synchronous Signal Averaging

The synchronous classification of signal averaging adds the additional constraint that each sample function must be initiated with respect to a specific trigger condition (often the magnitude and slope of the excitation). This means that the frequency response function can be formed as a summation of ratios of $X(\omega)$ divided by $F(\omega)$ since phase is preserved. Even so, the power spectra (least squares) approach is still the preferred FRF estimation method due to the reduction of variance and the usefulness of the ordinary coherence function. The ability to synchronize the initiation of digitization allows for the use of non-stationary or deterministic inputs with a resulting increased signal to noise ratio and reduced leakage. Both of these improvements in the frequency response function estimate are due to more of the input and output being observable in the limited time window.

The synchronization takes place as a function of a trigger signal occurring in the input (internally) or in some event related to the input (externally). An example of an internal trigger would be the case where an impulsive input is used to estimate the frequency response. All sample functions would be initiated when the input reached a certain amplitude and slope. A similar example of an external trigger would be the case where the impulsive excitation to a speaker is used to trigger the estimate of frequency response between two microphones in the sound field. Again, all sample functions would be initiated when the trigger signal reached a certain amplitude and slope.

### 2.3 Cyclic Signal Averaging

The cyclic classification of signal averaging involves the added constraint that the digitization is coherent between sample functions. This means that the exact time between each sample function is used to enhance the signal averaging process. Rather than trying to keep track of elapsed time between sample functions, the normal procedure is to allow no time to elapse between successive sample functions. This process can be described as a comb digital filter in the frequency domain with the teeth of the comb at frequency increments dependent upon the periodic nature of the sampling with respect to the event measured. The result is an attenuation of the spectrum between the teeth not possible with other forms of averaging.

This form of signal averaging is very useful for filtering periodic components from a noisy signal since the teeth of the filter are positioned at harmonics of the frequency of the sampling reference signal. This is of particular importance in applications where it is desirable to extract signals connected with various rotating members. This same form of signal averaging is particularly useful for reducing leakage during frequency response measurements and also has been used for evoked response measurements in biomedical studies.

A very common application of cyclic signal averaging is in the area of analysis of rotating structures. In such an application, the peaks of the comb filter are positioned to match the fundamental and harmonic frequencies of a particular rotating shaft or component. This is particularly powerful, since in one measurement it is possible to enhance all of the possible frequencies generated by the rotating member from a given data signal. With a zoom Fourier transform type of approach, one shaft frequency at a time can be examined depending upon the zoom power necessary to extract the shaft frequencies from the surrounding noise.

The application of cyclic averaging to the estimation of frequency response functions can be easily observed by noting the effects of cyclic averaging on a single frequency sinusoid. Figures 1 and 2...
represent the cyclic averaging of a sinusoid that is periodic with respect to the observation time period \( T \). Figures 3 and 4 represent the cyclic averaging of a sinusoid that is aperiodic with respect to the observation time period \( T \). By comparing Figure 2 to Figure 4, the attenuation of the nonperiodic signal can be clearly observed.

3. Theory of Cyclic Averaging

In the application of cyclic averaging to frequency response function estimates, the corresponding fundamental and harmonic frequencies that are enhanced are the frequencies that occur at the integer multiples of \( A f \). In this case, the spectra between each \( A f \) is reduced with an associated reduction of the bias error called leakage.

The first observation to be noted is the relationship between the Fourier transform of a history and the Fourier transform of a time shifted history. In the averaging case, each history will be of some finite time length \( T \) which is the observation period of the data. Note that this time period of observation \( T \) determines the fundamental frequency resolution \( A f \) via the Rayleigh Criteria \( (A f = \frac{1}{T}) \).

The Fourier transform of a history is given by:

\[
X(\omega) = \int x(t) e^{-i\omega t} \, dt
\]  

(1)

Using the time shift theorem of the Fourier transform, the Fourier transform of the same history that has been shifted in time by an amount \( t_0 \) is

\[
X(\omega) e^{-i\omega t_0} = \int x(t + t_0) e^{-i\omega t} \, dt
\]  

(2)

For the case of a discrete Fourier transform, each frequency in the spectra is assumed to be an integer multiple of the fundamental frequency \( A f = \frac{1}{T} \). Making this substitution in Equation (2) \((\omega = k \frac{2\pi}{T} \) with \( k \) as an integer) yields:

\[
X(\omega) e^{-i\frac{2\pi k}{T} t_0} = \int x(t + t_0) e^{-i\omega t} \, dt
\]  

(3)

Note that in Equation (3), the correction for the cases where \( t_0 = N T \) with \( N \) is an integer will be a unit magnitude with zero phase. Therefore, if each history that is cyclic averaged occurs at a time shift, with respect to the initial average, that is an integer multiple of the observation period \( T \), then the correction due to the time shift does not affect the frequency domain characteristics of the averaged result. All further discussion will assume that the time shift \( t_0 \) will be an integer multiple of the basic observation period \( T \).

The signal averaging algorithm for histories averaged with a boxcar or uniform window is:

\[
x(t) = \frac{1}{N_e} \sum_{n=0}^{N_e-1} x_i(t)
\]  

(4)
For the case where \( x(t) \) is continuous over the time period \( N_cT \), the complex Fourier coefficients of the cyclic averaged time history become:

\[
C_k = \frac{1}{T} \int_0^T x(t) e^{-j2\pi kT} dt
\]

(5)

\[
C_k = \frac{1}{N_c} \sum_{n=0}^{N_c-1} x(t) e^{-j2\pi kT} dt
\]

(6)

Finally:

\[
C_k = \frac{1}{N_cT} \sum_{n=0}^{N_c-1} x(t) e^{-j2\pi kT} dt
\]

(7)

Since \( x(t) \) is a continuous function, the sum of the integrals can be replaced with an integral evaluated from 0 to \( N_cT \) over the original function \( x(t) \). Therefore:

\[
C_k = \frac{1}{N_cT} \int_0^{N_cT} x(t) e^{-j2\pi kT} dt
\]

(8)

The above equation indicates that the Fourier coefficients of the cyclic averaged history (which are spaced at \( \Delta f = \frac{1}{T} \)) are the same Fourier coefficients from the original history (which are spaced at \( \Delta f = \frac{N_c}{T} \)). Note that the number of Fourier coefficients for the cyclic averaged history will be \( \frac{1}{N_c} \) the number of coefficients of the original history since the number and size of the frequency spacing changes by this factor. Also note that Parseval's Theorem, concerning the energy representation of each Fourier coefficient, is not preserved by the cyclic averaging process since the frequency information not related to the harmonics of \( \Delta f = \frac{1}{T} \) is removed [4].

The approach used to understand the frequency domain effects of windows on digital data can be used to understand the effect of cyclic averaging. Since cyclic averaging yields the Fourier coefficients of an effectively larger observation time \( (N_cT \) compared to \( T \)), the effect of cyclic averaging results in an effective frequency domain window characteristic that is a result of this longer observation time. However, the \( \Delta f \) axis needs to be adjusted to account for the actual frequencies that occur in the cyclic averaged spectra.

Figure 5 shows the two-sided frequency domain characteristic of the cyclic averaged \( (N_c = 4) \) case with a uniform window. Likewise, Figure 6 shows the two-sided frequency domain characteristic of the cyclic averaged \( (N_c = 4) \) case with a Hann window. Further detail of these characteristics is given in Figures 7 through 12. Figures 7 through 9 show the cyclic averaging effect in the frequency domain for the cases of 1.2 and 4 averages with a uniform window applied to the data. Figure 7 essentially represents no cyclic averaging and is the familiar characteristic of a uniform window [5-6]. Figures 8 and 9 show how cyclic averaging effects this window characteristic with respect to the \( \Delta f = \frac{1}{T} \) frequency spacing. Figures 10 through 12 show the cyclic averaging effect in the frequency domain for the cases of 1.2 and 4 averages with a Hann window applied to the original contiguous data. Figure 10 essentially represents no cyclic averaging and is the familiar characteristic of a Hann window [5-6]. Figures 11 and 12 show how cyclic averaging effects this window characteristic with respect to the \( \Delta f = \frac{1}{T} \) frequency spacing.

These figures demonstrate the effectiveness of cyclic averaging in rejecting nonharmonic frequencies. Practically, these figures also demonstrate that, based upon effectiveness or the limitations of the dynamic range of the measured data, a maximum of 16 to 32 averages is recommended. Realistically, 4 to 8 cyclic averages together with a Hann window provides a dramatic improvement in the FRP estimate.
**Figure 1. Uniform Window Characteristics**

**Figure 8. Cyclic Averaging \((N_c = 2)\) with Uniform Window**

**Figure 9. Cyclic Averaging \((N_c = 4)\) with Uniform Window**

**Figure 10. Ham Window Characteristics**

**Figure 11. Cyclic Averaging \((N_c = 2)\) with Ham Window**

**Figure 12. Cyclic Averaging \((N_c = 4)\) with Ham Window**
The results of cyclic averaging of a general random signal with the application of a uniform window are shown in Figures 13 and 14. Likewise, the results of cyclic averaging of a general random signal with the application of a Hann window are shown in Figures 15 and 16.

4. Practical Example

The implementation of cyclic signal averaging to frequency response function (FRF) estimation is not easily applicable to many existing discrete Fourier transform analyzers. The reason for this is that the user is not given control of the time data acquisition such that the cyclic averaging requirements can be met. However, many users currently are acquiring data with personal computer (PC) data acquisition boards or the VXI based data acquisition boards where control of the time data acquisition is more available to the user. In this environment, cyclic averaging is simple to implement by acquiring long data records and breaking the long data record into \( N_c \) contiguous time records which can be cyclic averaged.

The cyclic averaged inputs and outputs are normally computed by simply summing successive time records. The important requirement of the successive time records is that no data is lost. Therefore, these successive time records could be laid end to end to create the original longer time data record \( (N,T) \). The cyclic averaged records are then created by simply adding each time record of length \( T \) together in a block mode.

While the basic approach to cyclic averaging involves using the data weighted uniformly over the total sample time \( N,T \), the benefits that can be gained by using weighting functions can also be applied. The application of a Hann window to the successive time records before the summation occurs yields an even greater reduction of the bias error. Therefore, for frequency response function measurements, Hann windowed signal averaging should dramatically reduce the leakage errors which can exist when using broadband random excitation techniques to measure frequency response.

In Figure 17 through Figure 20, four different measurement cases are documented for the same FRF measurement. This data was acquired as typical data from a lightly damped, cantilever beam. Each figure shows the amplitude of an FRF with the associated ordinary coherence function shown as a measurement quality indicator. In each case, the \( H_1 \) FRF estimation algorithm was used:

- Case I: The FRF is computed from 64 asynchronous averages \( (N_a = 64) \). A uniform window (no additional window) is applied to the data. This is an unacceptable measurement and represents poor measurement procedure.
- Case II: The FRF is computed from 64 asynchronous averages \( (N_a = 64) \). A Hann window is applied to the data. This is a marginally acceptable measurement and represents a common measurement procedure.
- Case III: The FRF is computed from 16 cyclic averages \( (N_c = 4) \) and 16 asynchronous averages \( (N_a = 16) \). A uniform window (no additional window) is applied to the data. This is a marginally acceptable measurement and compares reasonably to Case II.
Case IV: The FRF is computed from 16 cyclic averages \( (N_c = 4) \) and 16 asynchronous averages \( (N_a = 16) \). A Ham window is applied to the data. This is a good measurement. Note particularly the increase in the FRF amplitude at the peak frequency locations compared to the three previous cases.

The value of \( N_c \) indicates the number of cyclic time records averaged together and \( N_a \) is the number of asynchronous auto and cross spectrum averages: a total of \( N_cN_a \) time records were sampled. This is done so that, statistically, the same amount of independent information is available in each averaging case. Note that in this example, the data for these cases was acquired only once. Each case results from processing the original time data differently.

Clearly, the measurement using cyclic averaging with the Hann window (Figure 20) shows a significant reduction of the bias error. An interesting point is that the data near the antiresonance is also drastically improved due to the sharp roll off of the Hann weighted averaging.

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Figure 17. Case I: Asynchronous Averaging

Figure 18. Case II: Asynchronous Averaging With Ham Window

Figure 19. Case III: Cyclic Averaging
5. Conclusions

cyclic averaging is a powerful digital signal processing tool that minimizes the leakage error when FRF measurements are being estimated. While existing discrete Fourier analyzers may not be able to include cyclic averaging for the FRF estimation case, computer-based data acquisition common to personal computer or workstation systems generally permit the user to apply cyclic averaging together with asynchronous or synchronous averaging to effectively minimize both random errors and the leakage bias error.

6. Acknowledgements

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7. References