ABSTRACT. Source identification problems are often solved by the multiple-input and single-output modeling. Several techniques are presently available to tackle the case where the measured inputs are correlated to each other. One is to condition the measured inputs using partial coherence functions. In this case, priority among the inputs must be determined somehow before conditioning. Another is to decompose the measured inputs into the virtual ones which are independent of each other and then to deal with the virtual inputs in the MISO analysis.

In this method, priority determination is extremely crucial and, hence, several strategies have been recommended so far. One is to put the inputs in the order of the magnitude of ordinary coherence function between each input and the output, which looks simple to apply and reasonable from physical points of view but has no philosophical reasoning. Another is to check causality of the correlated inputs, which was introduced by the authors. The causality of the correlated inputs is checked by investigating impulse response functions between two relevant inputs in the negative time region. In this paper, an alternative to the second approach is introduced where Hilbert transforms of the frequency response functions between the two correlated inputs are utilized. Theoretical backgrounds of this method are presented together with some results from applications to the analysis of an acoustical system.

1. Introduction

Source identification involves various problems such as selection of physically meaningful number and locations of inputs, estimation therefrom of independent source number, and behavior and contribution of each source. Thus the source identification itself is something more than simple data processing of the multiple-input and single-output (MISO) measurements.

The inputs must be measured at locations as close as possible to what are believed as real sources. If the measured inputs are not correlated at all to each other, contribution of each input to the output can be obtained without any difficulty from the frequency response and ordinary coherence functions between the inputs and output. On the other hand, if the measured inputs are partially correlated to each other to some extent, the frequency response function approach cannot be directly applied. One solution to the latter case is to obtain a new set of uncorrelated inputs by conditioning the original correlated inputs, which is called partial coherence function approach and will be further discussed later in this section. Another solution is to decompose the correlated real inputs into the virtual ones in such a way that they may be independent of each other and then to compute contribution of each virtual input.

In the partial coherence function approach, the most critical procedure is how to determine priority among the correlated inputs. One recommendation is to put the inputs in the order of magnitudes of ordinary coherence functions with the output. This is very simple to apply and looks reasonable from physical points of view but has no philosophical reasoning. Another method is to check causality between any pairs of mutually correlated inputs. The causality of the correlated inputs is determined by investigating a pair of impulse response functions in the negative time region, which can be easily obtained by inverse Fourier transform of the frequency response functions between the two correlated inputs. That is, the priority is given to one signal out of the pair which yields the impulse response in the negative time.
domain closer to zero when it is taken as the input and the other as the output.

In this paper, another method to determine the causality of the correlated inputs is presented which uses Hilbert transforms of the frequency response functions between the correlated pair. The method is based on the analyticity of a transfer function if it represents a physically realizable or causal linear system. The analyticity of a transfer function can be expressed by the Hilbert-transform pairness of the real and imaginary parts of the frequency response functions. Theoretical background of this approach is presented. The method is applied to the analysis of an acoustical system and the results are compared with those by the impulse response function technique[6].

2. Basic Background

The Cauchy's integral theorem[1]:

$$\oint_{C} G(s) ds = 0 \quad (1)$$

where $G(s)$ is an analytic function of complex variable $s$ inside and on a contour $C$, is a good point to start with to understand the Hilbert transform approach.

Let the function $G(s)$ be a transfer function of a linear system with poles in the left half plane of the complex $s$-domain or analytic in the right half plane. Consider another function $G(s)/(s-j\omega_0)$, which is also analytic in the right half plane. Applying the Cauchy's integral theorem to this new function along a contour $C(I-II-III)$ as shown in Figure 1(a) gives:

$$\int_{C} \frac{G(s)}{s-j\omega_0} \; ds + \int_{II} \frac{G(s)}{s-j\omega_0} \; ds + \int_{III} \frac{G(s)}{s-j\omega_0} \; ds = 0 \quad (2)$$

The first term becomes zero with increase of the radius of the contour to infinity as long as $G(s)$ goes to zero for infinite $s$. The second term can be rewritten as a line integral with the increase of contour size to infinity:

$$\int_{II} \frac{G(s)}{s-j\omega_0} \; ds = - \int_{\omega=-\infty}^{\omega=\infty} \frac{G(\omega)}{j\omega_0} \; d\omega \quad (3)$$

The third term can be evaluated using the Cauchy's principal value:

$$\int_{III} \frac{G(s)}{s-j\omega_0} \; ds = -j\pi G(j\omega_0) \quad (4')$$

Therefore, Eq.(2) can be rewritten as follows:

$$G(j\omega_0) = -\frac{1}{j\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{G(\omega)}{\omega-j\omega_0} \; d\omega \quad (5)$$

Defining the real and imaginary parts of the complex function $G(j\omega)$ by $R(\omega)$ and $X(\omega)$ respectively yields the righthand side of Eq.(5) in the following form:

$$\begin{align*}
-\frac{1}{j\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{R(\omega)}{\omega-j\omega_0} \; d\omega + \frac{1}{\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{X(\omega)}{\omega-j\omega_0} \; d\omega
\end{align*} \quad (6)$$

Since real and imaginary parts of the lefthand side $G(j\omega_0)$ in Eq.(5) can be represented by $R(\omega_0)$ and $X(\omega_0)$ respectively, they are related to $X(\omega)$ and $R(\omega)$ by:

$$\begin{align*}
R(\omega_0) &= -\frac{1}{\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{X(\omega)}{\omega-j\omega_0} \; d\omega \quad (7.a) \\
X(\omega_0) &= \frac{1}{\pi} \int_{\omega=-\infty}^{\omega=\infty} \frac{R(\omega)}{\omega-j\omega_0} \; d\omega \quad (7.b)
\end{align*}$$

which are the very definitions of the Hilbert forward and backward transform of two real functions $R(\omega)$ and $X(\omega)$.

Next, consider another contour integral:

$$g(t) = \oint_{C} G(s) \exp(st) \; ds \quad (8)$$

where the contour can be taken anywhere in the left half plane for $t > 0$ and in the right plane for $t < 0$. Noting that $G(s)$ is analytic in the right half plane and that $g_0$ is $G(s)\exp(st)$, the contour integral along $C_1$ in Figure 1(b) must vanish according to the Cauchy's integral theorem Eq.(1). Since $G(s)$ is not analytic in the left half plane but has poles, the contour integral along $C_2$ in Figure 1(b) can be derived using residue theorem:

$$g(t) = \oint_{C} G(s) \exp(st) \; ds = - \sum_{n} j2\pi G(s_n) \quad t > 0 \quad (9)$$

The contour integral consists of a line integral along the imaginary axis and an integral along the half circle, which goes to zero with increase of the radius as long as $G(s)$ goes to zero for infinite $s$. Therefore, Eq.(8) can be rewritten as:

1238
which is the very definition of inverse Fourier transform of the frequency response function \( G(j\omega) \). Therefore, if real and imaginary parts of the frequency response function \( G(j\omega) \) satisfies Hilbert transform pairness, its inverse Fourier transform \( g(t) \) must be a causal function which is zero in the region of \( t < 0 \).

Two frequency response functions can be defined between any pair of correlated inputs to the MISO system, e.g., \( x_1(t) \) and \( x_2(t) \), by interchanging the order of computation, that is, \( G_{12}(j\omega) = X_1(j\omega)/X_2(j\omega) \) and \( G_{21}(j\omega) = X_2(j\omega)/X_1(j\omega) \). If real and imaginary parts of \( G_{12}(j\omega) \) satisfies Eq.(7) over a given frequency range, then the input 1 would have caused the input 2 over that specific frequency range, and vice versa.

### 3. Computation of Hilbert transform

If \( G(j\omega) \) is Fourier transform of a real signal, then its real part \( R(\omega) \) is an even function and imaginary part \( X(\omega) \) is an odd function. Hence convolution integrals in Eq.(7) can be rewritten:

\[
R(\omega) = -\frac{2}{\pi} \int_0^\infty \frac{\omega X(\omega) d\omega}{\omega^2 - \omega_0^2} \tag{10.a}
\]

\[
X(\omega) = \frac{2\omega}{\pi} \int_0^\infty \frac{R(\omega) d\omega}{\omega^2 - \omega_0^2} \tag{10.b}
\]

For a set of discrete date, convolution summation given below replaces the convolution integral:\[5.\]

\[
R(\omega_j) = -\sum_{k=1}^n \frac{X(\omega_j) \omega_k \Delta \omega}{\omega_k^2 - \omega_j^2} \tag{11.a}
\]

\[
X(\omega_j) = \frac{2\omega_j}{\pi} \sum_{k=1}^n \frac{R(\omega_k) \Delta \omega}{\omega_k^2 - \omega_j^2} \tag{11.b}
\]

The convolution integral can be alternatively represented by using Fourier transform and signum function \( \text{sgn}(t) \) as follows:

\[
H[R(\omega)] = F^{-1}[F\{R(\omega)\} \cdot \text{sgn}(t)] \tag{12}
\]

where \( \text{sgn}(t) \) is either 1 or -1 depending on sign of \( t \) as shown in Figure 2(a). Since frequency response functions are mostly obtained at discrete points and the discrete fast Fourier transform is computationally far more efficient than the convolution summation.

Hilbert transform is usually obtained by adapting discrete version of Eq.(12) rather than Eq.(11) to discretized data since the computational cost. Noting that, however, the forward Fourier transform on the right-hand side of Eq.(12) is taken for finite length of data, the signum function acts as a window function as shown in Figure 2(b) on the inverse Fourier transform of \( R(\omega) \).

It is well known that a window function which does not decay to zero as in Figure 2(b) at the ends of the window will bring about so called leakage problem. To reduce the leakage, a modified signum function can be adapted. One example is to replace the Fourier transform of \( \text{sgn}(t) = j/\pi \omega \), with a discrete function, \( U(k) \), that would not have discontinuities. The modified signum function shown in Figure 2(c) is obtained from such an example defined by:

\[
U(k) = \begin{cases} 
\frac{-j}{2\pi k} & \text{for } k = 1 \\
\frac{j}{\pi k} & \text{for } 2 \leq k < n \\
0 & \text{for } k = 0, k = n \\
\frac{-j}{\pi(k-n)} & \text{for } n+1 \leq k \leq 2n-2 \\
\frac{j}{2\pi k} & \text{for } k = 2n = 1 
\end{cases} \tag{13}
\]

where \( n \) is the total number of frequency points.

### 4. Experiment and Results

A three-input and single-output acoustic system shown in Figure 3, which was used in reference\[6] to check the causality of correlated inputs by impulse response function approach, is reused in this study. The signals were generated from three independent random noise sources and then fed into circuits of filtering and summation before driving the loudspeakers. The input signals were measured near each loudspeaker by three microphones.

Power of the input signals are concentrated in band-limited frequency regions as shown in Figure 4(a), (b) and (c) and they are partially interrelated as shown in Figure 5. Thus, first of all, priority among the inputs must be determined in frequency ranges over which the coherence is significantly large in order to apply the partial coherence function approach.

Let us consider as an example a pair of correlated signals input 2 and input 3. Two frequency response functions, \( G_{21}(j\omega) \) and \( G_{22}(j\omega) \), between the two inputs can be easily obtained. Figure 6 shows...
two impulse response functions, \( g_{2a}(t) \) and \( g_{3a}(t) \), obtained simply from inverse Fourier transform of \( G_{2a}(j\omega) \) and \( G_{3a}(j\omega) \) over 0 to 4 kHz. It is difficult to tell from these figures which one causes the other because the level of the impulse response functions in the negative time region is very high in both cases. This difficulty could be resolved by dividing the frequency range of analysis in previous study[7].

Because, however, the Hilbert transform of real and imaginary parts of the frequency response function illustrate the information of causality directly in frequency domain as shown in Figure 7, it is more efficient to determine the priority over specific frequency range of interest. The solid and dotted lines represent imaginary parts and Hilbert transform of real parts respectively. If the two curves match each other well over a given frequency range in one plot \( G_{2a}(j\omega) \), but not in the other plot \( G_{3a}(j\omega) \), then it means that the input 2 causes the input 3 over that specific frequency range. For example, the two curves in Figure 7 match very well in \( G_{3a}(j\omega) \) around 1.5 kHz while they do so in \( G_{2a}(j\omega) \) around 1 kHz and 3 kHz. That is, input 2 causes input 3 around 1.5 kHz and input 3 causes input 2 around 1 kHz and 3 kHz.

The results in Figure 7 can be made clearer by observing the Hilbert transform pairs in selected frequency range after band pass filtering to two inputs, as shown in Figures 8 and 9, so that the dynamic range of the presentation may be maximized. From these figures, we can narrow down the frequency range; input 2 causes input 3 over 1.3 to 1.6 kHz while input 3 causes input 2 over 2.7 to 3.2 kHz.

Figure 10 shows two impulse response functions obtained from inverse Fourier transform of \( G_{2a}(j\omega) \) and \( G_{3a}(j\omega) \) corresponding to each selected frequency in Figure 8 and 9. The causality of \( h_{2a}(t) \) is related to the exciting signal via filter \( L_{2a}(j\omega) \) with delay time, 6.7 msec, that is shown at the peak in Figure 10(b). But the delay time, 3.9 msec, in Figure 10(b) is related to the path difference of sound wave propagation from loudspeaker 3 to each microphone 2 and 3.

5. Conclusions

The basic concept of the causality checking technique using Hilbert transform is presented to determine the priority among a pair of correlated inputs, which is essential in source identification via MISO modeling. The Hilbert transform pairness of real and imaginary parts of a frequency response function which has poles in the right half s-plane is shown mathematically to be equivalent to the causality of the impulse response function estimated from the frequency response function. Therefore, basically either of the Hilbert transform and the impulse response function approach can be applied to determine the priority in a pair of correlated inputs. In a case where the priority changes with frequency, however, the impulse response function approach needs care in selecting the specific frequency range of interest because the inverse Fourier transform of the frequency response function over all frequency components will mix up the causality information. That is, in the Hilbert transform approach, variation of the causality with frequency can be more easily checked over a wide frequency range.

References

Figure 1. Integral contour in complex $s = \sigma + j\omega$ plane. (a) for Hilbert transform; (b) for inverse Fourier integral.

Figure 2. Signum function. (a) Ideal signum function; (b) Periodic signum function; (c) Modified signum function.

Figure 3. Experimental setup of three-input and single-output acoustical system.
Figure 4. Power spectrum of inputs and output
S_{11}(f); (b) S_{22}(f); (c) S_{33}(f); (d) S_{yy}(f).

Figure 5. Coherence function between correlated inputs. (a) \gamma_{12}^2(f); (b) \gamma_{13}^2(f); (c) \gamma_{23}^2(f).

Figure 6. Impulse response function between input 2 and input 3 over 0 to 4 kHz. (a) g_{32}(t); (b) g_{23}(t).
Figure 7. Imaginary parts and Hilbert transform of real parts between input 2 and input 3 over 0 to 4 kHz. (a) $G_{32}(f)$; (b) $G_{23}(f)$.

Figure 8. Imaginary parts and Hilbert transform of real parts between input 2 and input 3 over 1.3 to 1.6 kHz. (a) $G_{32}(f)$; (b) $G_{23}(f)$.

Figure 9. Imaginary parts and Hilbert transform of real parts between input 2 and input 3 over 2.7 to 3.2 kHz. (a) $X(\omega)$, $H[R(f)]$. (b) $G_{32}(f)$; (b) $G_{23}(f)$.

Figure 10. Impulse response function between input 2 and input 3. (a) $g_{32}(t)$ over 1.3 to 1.6 kHz; (b) $g_{23}(t)$ over 2.7 to 3.2 kHz.