Nonlinear parametric identification of an rubber coupling using vibration test and matrix-exponential method

Zhiqiang Shi, J. Richard Houghton

Machine Dynamics Research Laboratory
Dept. of Mechanical Engineering
Tennessee Technological University
Cookeville, TN 38505

Abstract

A numerical method for estimating the nonlinear characteristics of a coupling connecting two cantilever beams is presented in this paper. Nonlinearity representation of the coupling was assumed. The parametric values were determined by optimum matching the numerical solution of system response with experimental measurement. Pseudo mode shape was introduced to transform the governing motion equations to ordinary differential equations, through which the structure dynamic response was solved using matrix-exponential method. A Simplex multivariable optimization scheme was used to implement the parametric estimations. The proposed method shows potential in solving similar parametric estimation problems.

Nomenclature

$w(x,t)$: displacement of the cantilever beam
$\rho$: mass density
$A$: cross section area of the beam
$E$: Young’s modules
$I$: area moment
$F(t)$: impact force
$I_c$: impact location
$f(t)$: rubber connection force
$l_c$: rubber connection location
$c$: damping coefficient of the rubber
$k$: stiffness coefficient of the rubber
$\tilde{w}(t)$: experimental acceleration
$\tilde{\Phi}(x)$: mode shape

1 Introduction

The design and control of large complex structures have aroused tremendous interests in system identification and parameter estimation of mechanical systems in recent decades [1-7]. One of the main issues in studying the behavior of such structures is the estimation of structural parameters, especially the stiffness and damping mechanism [3-4]. Parameter estimation is an inverse dynamic problem: find a set of parameters such that the numerical results calculated from analytical modal match the experimental data subject to certain criteria. Both displacement, velocity and acceleration signals can be used to perform the estimation task. The damping and vibratory effects of a beam with various frictional supports had been studied using a single spatial model and first order harmonic balance method (Ferri, 1992). A finite dimensional approximation method utilizing the Ritz-Galerkin technique has been used to estimate the dynamic properties, damping, stiffness of the end mass, and the visco-elastic structural damping (Banks, 1988). Shen and Taylor (1993) had solved the Timoshenko equations analytically, and estimated the coefficients of the partial differential equations, based on the optimal matching between measurements and theoretical calculations.

In studying distributed parameter systems, such factors as coupling of interconnected structures, clearance at the joint, unbalanced mass distribution, the inherent nonlinearities of materials etc. are all prone to bring out nonlinearity in the system, which will result in significant differences from the simplified linear model. Several examples had shown that neglecting the nonlinear effects tends to cause unsatisfactory results in system dynamics. In such cases the nonlinear phenomena plays an important role and must be identified and accounted for. Methods for studying nonlinear problems can be
classified as nonparametric and parametric. The nonparametric approach intends to identify the functional mechanism of a nonlinear effect, while the parametric approach tries to determine specifically the values of a assumed nonlinear representation.

In this paper, a structure that consists of two cantilever beams connected by a rubber was studied. The nonlinear dynamic characteristic of the rubber, in terms of damping and stiffness, was presumed. The parametric values were determined by optimum fitting of numerical solution of the system response to the experimental measurements using a multivariable optimization algorithm. The structure deformation was assumed as the sum of several modes, which were adequately separated in frequency domain. Pseudo mode shapes were assumed to transform the governing partial differential equations into ordinary differential equations. The corresponding dynamic response at the measurement point was calculated using the matrix-exponential method. Parametric values of stiffness and damping were determined by matching the numerical solution of system dynamic response to the experimental measurements in the time domain.

2 Theoretical formulation of the problem

The structure studied in this paper consists of two cantilever beams that are coupled through a rubber, of which the nonlinear behavior is to be analyzed. The experiment setup is shown in Figure 1.

Let \( w_1(x,t) \) and \( w_2(x,t) \) denote the transverse vibration of the two cantilever beams respectively, then the motion equations are given by Euler-Bernoulli theory as:

\[
\frac{\partial^4 w_i(x,t)}{\partial t^2} + EI \frac{\partial^2 w_i(x,t)}{\partial x^4} = F_i(t_i,t) \cdot f(t_i,t)
\]

where \( F_i(t_i,t) \) is an impulse excitation applied to the structure at \( x = l_i \), and \( f(t_i,t) \) is the coupling force introduced by the rubber connection at \( x = l_i \), which is

\[
\phi \frac{\partial^2 w_2(x,t)}{\partial t^2} + EI \frac{\partial^2 w_1(x,t)}{\partial x^4} = -f(t_i,t)
\]

modeled as:

\[
\sum_{k=1} \phi \left( w_i(t_0,t) \cdot w_i(t_0,t) \right) \cdot f_k \left( w_i(t_0,t) \cdot w_i(t_0,t) \right)
\]

For the subscripts in equation (2.3), \( i = 1 \) denotes the linear viscous damping and linear stiffness, and \( i = 2 \) denotes the nonlinear quadratic damping and stiffness. \( c_i \) and \( k_i \) (\( i = 1,2 \)) are the coefficients to be determined.

Initial and boundary conditions are given in equations (2.4-2.5) and (2.6-2.9) respectively:

\[
w_1(x,0) = 0 \quad (2.4)
\]

\[
\frac{\partial w_1(x,0)}{\partial t} = 0 \quad (2.5)
\]

\[
w_1(0,t) = 0 \quad (2.6)
\]

\[
\frac{\partial w_1(0,t)}{\partial x} = 0 \quad (2.7)
\]

\[
\frac{\partial^2 w_1(t,t)}{\partial x^2} = 0 \quad (2.8)
\]

\[
\frac{\partial^2 w_1(t,t)}{\partial x^3} = 0 \quad (2.9)
\]

\( i = 1,2 \). The dynamic of the structure is fully defined by equations (2.1-2.9).

3 Numerical solution of dynamic response

3.1 Matrix-Exponential Method

In this paper, the dynamic response of the structure was solved using matrix-exponential method. This method is based on the principle of variation of parameters. A brief introduction of matrix-exponential method is given below. More details can be found in [7].

Consider the first order differential equation:

\[
y(t) = ay(t) + p(t)
\]

where \( a \) is constant and \( p(t) \) is the excitation. Assume the solution of equation (3.1) is of the form:

\[
y(t) = c(t) e^{at}
\]

where \( c(t) \) is an unknown function of time. Differentiating
equation (3.2) with respect to time, then substitute into equation (3.1), using integration and evaluating from \( t=0 \), yields the solution:

\[
y(t) = y_0 e^{-\omega t} + \int_0^t e^{-\omega(t-t')} p(t') dt'
\]  

(3.3)

Matrix-Exponential method is a discrete form of equation (3.3) by considering the response \( y(t) \) in the interval \( t_0 + h \), where \( h \) is step size. Furthermore, assuming the input \( p(t) \) is constant \( p_0 \) in this interval (zero order hold method), then yields the following:

\[
y(h) = y_0 e^{-\omega h} + p_0 e^{-\omega h} \int_0^h e^{\omega \tau} d\tau
\]  

(3.4)

\[
y_0 = y_0 e^{-\omega h} + \frac{p_0 e^{-\omega h}}{\omega}
\]  

(3.5)

Note the only approximation introduced in deriving equation (3.4) is that excitation \( p(t) \) is assumed to be constant across each interval. When \( h \) is small enough, this assumption is well justified. Also note that exponential term \( e^{\omega h} \) is constant for a fixed step size. Therefore equation (3.4) can be further reduced to the iterative form:

\[
y_{n+1} = y_n e^{-\omega h} + \frac{p_0 e^{-\omega h}}{\omega}
\]  

where \( u = e^{\omega h} \) and \( v = (u-1)/\omega \)

Higher order differential equations can also be solved using this method by introducing the Hamilton’s canonical form, which simplifies the equation into a set of first order differential equations.

3.2 Application of Matrix-Exponential Method

To transform the governing partial differential equations (2.1-2.2) into ordinary differential equations, a pseudo mode shape was introduced around each mode of the structure. The basis of this assumption was that the structural modes were well separated in frequency. The flexural beam displacement \( w(x,t) \) was represented by the mode \( \phi_j(x) \) and modal amplitude \( z_j(t) \):

\[
w(x,t) = \phi_j(x)z_j(t)
\]  

(3.6)

Substituting equation (3.6) into equations (2.1-2.2) and performing Galerkin procedure \([2]\) yields:

\[
\begin{align*}
\dot{z}_j(t) & = -\frac{1}{m} \int_0^L \phi_j(x) \phi''(x) dx - \int_0^L E I \phi''(x) \phi'(x) dx \\
& - \int_0^L F(x,0) \delta(x-L_j) \phi(x) dx + \int_0^L \int_0^L \phi_j(x) \phi_j(x) dx + \int_0^L K \phi_j(x) \phi_j(x) dx \\
& = -\frac{1}{m} \int_0^L \phi_j(x) \phi''(x) dx - \int_0^L E I \phi''(x) \phi'(x) dx \\
& - \int_0^L F(x,0) \delta(x-L_j) \phi(x) dx + \int_0^L \int_0^L \phi_j(x) \phi_j(x) dx + \int_0^L K \phi_j(x) \phi_j(x) dx
\end{align*}
\]  

(3.7)

Substitute the explicit expression of \( f(t) \) from equation (2.3), equations (3.7-3.8) can be written in canonical form as:

\[
z_j(t) = -\frac{1}{m} \int_0^L \phi_j(x) \phi''(x) dx - \int_0^L E I \phi''(x) \phi'(x) dx \\
- \int_0^L F(x,0) \delta(x-L_j) \phi(x) dx + \int_0^L \int_0^L \phi_j(x) \phi_j(x) dx + \int_0^L K \phi_j(x) \phi_j(x) dx
\]  

(3.8)

For a cantilever beam, the mode shape \( \phi_j(x) \) is given as:

\[
\phi_j(x) = \cos \theta \phi_j(x) - \sin \theta \phi_j(x)
\]  

(3.13)

In this paper, only the first two modes of the structure are considered. When \( N=1 \) and substitute equation (3.13) into equations (3.11-12) yields the governing motion equation for the first mode, and \( N=2 \) corresponds to the second mode.

To use matrix-exponential method solve the 2nd order differential equations of (3.11-12), they are written in Hamilton’s canonical form by defining the state space vector:

\[
\dot{\mathbf{y}} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \\ y_5(t) \\ y_6(t) \end{bmatrix} = \begin{bmatrix} y_1(t) - z_1(t) \\ y_2(t) - z_2(t) \\ y_3(t) - z_3(t) \\ y_4(t) - z_4(t) \end{bmatrix}
\]  

(3.14)
4 Optimization scheme for estimating the dynamic characteristics of the coupling

Dynamic responses of the structure were obtained experimentally when the structure was excited by an impulse force. The acceleration signals were recorded for $t < t_i$ at $x = 1$. The numerical solution was obtained by solving equation (3.15) using matrix-exponential method, subject to a set of parameter $q$. The objective function $J(q)$ is defined as follows:

$$J(q) = \int_0^t \left[ \frac{\partial^2 w_l(t, t)}{\partial t^2} - w_g(t, t) \right]^2 dt$$

where $w_l(t, t)$ and $w_g(t, t)$ denote the experimental acceleration signals.

The objective function $J(q)$ is defined as follows:

$$J(q) = \int_0^t \left[ w_l(t, t) - w_g(t, t) \right]^2 dt$$

where $w_l(t, t)$ and $w_g(t, t)$ denote the experimental acceleration signals.

The unknown parameters were solved by minimizing equation (4.1), which would give an optimum match of the numerical calculation and experimental measurements in the least mean square sense. Then this set of $q(c_1, c_2, k_1, k_2, k_3)$ and equation (2.3) give an explicit representation of the dynamics characteristic of the rubber coupling.

The experimental signal was filtered away resonances of the assumed pseudo mode through a digital filter. The transient part of the signal was discarded as it did not represent the system dynamic characteristics of interest. The procedure for calculating the dynamic response of the system under the assumed mode shape is an iterative process. A MATLAB program FMINS utilizing Simplex algorithm for multivariable optimization purposes is utilized to find the set of $c, k$ that will give the best match between experiment measurement and numerical solutions.

5 Experiment application

5.1 Analysis of single beam

The beam studied has a cross section of $3.175 \times 25.4 \text{mm}$. The length of cantilever part is 0.4953m. Other physical dimensions are listed in Table 1. Theoretical values of the first and second modal frequencies were calculated as 10.55Hz and 66.13Hz respectively.

<table>
<thead>
<tr>
<th>I (m^4)</th>
<th>E (pa)</th>
<th>$\rho$ (kg/m^3)</th>
<th>A (m^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.75 \times 10^{-12}</td>
<td>2.08 \times 10^{11}</td>
<td>7.85 \times 10^{2}</td>
<td>8.065 \times 10^{5}</td>
</tr>
</tbody>
</table>

The dynamic response of a single beam was displayed in Figure 2 in both time and frequency domain. The first two mode frequencies were identified as 10.16Hz and 66.28Hz.

The motion equation of the beam at the measurement point subject to an impact excitation is given as:

$$\ddot{q}(t) + c \dot{q}(t) + kq(t) = F(t)\delta(t)$$

where $c$ and $k$ are viscous damping and linear stiffness respectively. Equation (5.1) can be written in canonical form as:

$$\ddot{y}(t) - [0 \ 1] \begin{bmatrix} \psi(t) \\ \dot{\psi}(t) \end{bmatrix} * \begin{bmatrix} F(t) \delta(t) \end{bmatrix}$$

The linear stiffness and viscous damping were estimated for each mode through the procedures developed in the previous section. The transient of experimental signal was discarded and the signal was filtered at the two identified modal frequencies. An initial value of $c$ and $k$ were selected to calculate the numerical solution of equation (5.2) using matrix-exponential method. The $c$ and $k$ were optimized by FMINS, a multivariable

\[4\] Optimization scheme for estimating the dynamic characteristics of the coupling

Dynamic responses of the structure were obtained experimentally when the structure was excited by an impulse force. The acceleration signals were recorded for $t < t_i$ at $x = 1$. The numerical solution was obtained by solving equation (3.15) using matrix-exponential method, subject to a set of parameter $q$. The objective function $J(q)$ is defined as follows:

$$J(q) = \int_0^t \left[ w_l(t, t) - w_g(t, t) \right]^2 dt$$

where $w_l(t, t)$ and $w_g(t, t)$ denote the experimental acceleration signals.

The unknown parameters were solved by minimizing equation (4.1), which would give an optimum match of the numerical calculation and experimental measurements in the least mean square sense. Then this set of $q(c_1, c_2, k_1, k_2, k_3)$ and equation (2.3) give an explicit representation of the dynamics characteristic of the rubber coupling.

The experimental signal was filtered away resonances of the assumed pseudo mode through a digital filter. The transient part of the signal was discarded as it did not represent the system dynamic characteristics of interest. The procedure for calculating the dynamic response of the system under the assumed mode shape is an iterative process. A MATLAB program FMINS utilizing Simplex algorithm for multivariable optimization purposes is utilized to find the set of $c, k$ that will give the best match between experiment measurement and numerical solutions.

5 Experiment application

5.1 Analysis of single beam

The beam studied has a cross section of $3.175 \times 25.4 \text{mm}$. The length of cantilever part is 0.4953m. Other physical dimensions are listed in Table 1. Theoretical values of the first and second modal frequencies were calculated as 10.55Hz and 66.13Hz respectively.

<table>
<thead>
<tr>
<th>I (m^4)</th>
<th>E (pa)</th>
<th>$\rho$ (kg/m^3)</th>
<th>A (m^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.75 \times 10^{-12}</td>
<td>2.08 \times 10^{11}</td>
<td>7.85 \times 10^{2}</td>
<td>8.065 \times 10^{5}</td>
</tr>
</tbody>
</table>

The dynamic response of a single beam was displayed in Figure 2 in both time and frequency domain. The first two mode frequencies were identified as 10.16Hz and 66.28Hz.

The motion equation of the beam at the measurement point subject to an impact excitation is given as:

$$\ddot{q}(t) + c \dot{q}(t) + kq(t) = F(t)\delta(t)$$

where $c$ and $k$ are viscous damping and linear stiffness respectively. Equation (5.1) can be written in canonical form as:

$$\ddot{y}(t) - [0 \ 1] \begin{bmatrix} \psi(t) \\ \dot{\psi}(t) \end{bmatrix} * \begin{bmatrix} F(t) \delta(t) \end{bmatrix}$$

The linear stiffness and viscous damping were estimated for each mode through the procedures developed in the previous section. The transient of experimental signal was discarded and the signal was filtered at the two identified modal frequencies. An initial value of $c$ and $k$ were selected to calculate the numerical solution of equation (5.2) using matrix-exponential method. The $c$ and $k$ were optimized by FMINS, a multivariable...
optimization scheme, to tit the filtered experimental signal at each mode.

The time response of first frequency component (1st mode) and the entire experiment signal (1st and 2nd mode) and numerical results are shown in Figure 3 and Figure 4 respectively.

![Figure 3 Time response of the 1st frequency component of the single cantilever beam](image)

![Figure 4 Time response of the single cantilever beam summation of 1st and 2nd frequency component](image)

Thus the stiffness and damping are determined, which will be used as initial values in the analysis of coupled structure. The result is listed in Table 2.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Damping (N/m/s)</th>
<th>Stiffness (N/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st mode</td>
<td>0.1633</td>
<td>423 I</td>
</tr>
<tr>
<td>2nd mode</td>
<td>0.8799</td>
<td>174170</td>
</tr>
</tbody>
</table>

It is noted that the damping quantities for higher mode is significantly larger than that of the lower mode.

The initial value of $c$ and $k$ is important in obtaining convergence in the optimization process. As the dynamic response of the structure is partly known, i.e. the frequency spectrum, and with previous knowledge about the structure, the initial value can be carefully chosen to give a convergent calculation.

5.2 Analysis of the coupled structure

The dynamic response of the coupled structure and its frequency spectrum is shown in Figure 5.

![Figure 5 Time response of the coupled structure and its frequency spectrum](image)
and Figure 7 respectively.

**Figure 6** Time response of the 1st frequency component of the coupled structure: - experimental; I.: numerical fitting

**Figure 7** Time response of the coupled structure summation of 1st and 2nd frequency component: - experimental; I.: numerical fitting

The estimated damping coefficient and stiffness of the rubber coupling are listed in Table 3.

<table>
<thead>
<tr>
<th>Mode</th>
<th>( c_1 ) (N/m/s)</th>
<th>( c_2 ) (N/m/s)</th>
<th>( k_1 ) (N/m)</th>
<th>( k_2 ) (N/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>0.287</td>
<td>1.347</td>
<td>0.408</td>
<td>50.0</td>
</tr>
<tr>
<td>2nd</td>
<td>0.000</td>
<td>22.221</td>
<td>0.429</td>
<td>57.0</td>
</tr>
</tbody>
</table>

From the estimated parameters, it can be seen the nonlinear part of the damping and stiffness does not change significantly. The increase of the square damping of 2nd mode corresponds to the fact that higher frequency component dies out more quickly.

The identified damping and stiffness coefficients listed in Table 3, together with equation (2.3) give an explicit representation of the dynamic property of the connection rubber.

Also observed is that due to the rubber connection, the damping and stiffness of the cantilever beam itself show some changes as indicated in Table 4.

<table>
<thead>
<tr>
<th>Mode</th>
<th>damping (N/m/s)</th>
<th>stiffness (N/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st mode</td>
<td>1.598</td>
<td>4322</td>
</tr>
<tr>
<td>2nd mode</td>
<td>2.122</td>
<td>174170</td>
</tr>
</tbody>
</table>

The stiffness value of the coupled structure is about the same as the singled case. While the damping factor show much changes for both modes. This suggests that in studying coupled structures the changes of the dynamic property of each substructure, especially the damping factor should be well understood to obtain an accurate model of the whole structure.

### 6 Conclusion

A numerical method for estimating the stiffness and damping coefficient of a cantilever beam as well as the nonlinear characteristics of a coupling connecting two cantilever beams is presented in this paper. The parametric estimation algorithm based on matrix-exponential method gives a satisfactory matching of numerical calculation and the experiment. To improve the accuracy of parametric estimation, high order hold method for the matrix exponential formulation could be used. The estimated parameter, together with the assumed nonlinearity formation provides a reasonable representation of the element dynamic behavior. The same approach shows promise in solving other parametric estimation problems.

### References:


