A new model error indicator function based on singular value decomposition (SVD) and QR permutation decomposition techniques is proposed. Since an updating problem including large numbers of updating parameters is usually ill-conditioned, a singular value decomposition technique is first used to determine the meaningful sub-matrix of the system data matrix. A QR permutation decomposition with column pivoting is then performed on this sub-matrix to find the permuted parameters. The equation error norms corresponding to sequential reduced systems are calculated and an error termination criterion is applied to determine the number of meaningful updating parameters. The proposed method is compared with two other popular error indicator functions. Test case includes a free-free supported beam structure.

**NOMENCLATURE**

- \([A],[A_{perm}]\) original and permuted data matrices
- \([A'],[A^*]\) selected and unselected data sub-matrices
- \([H]\) observation vector
- \([C],[C_A]\) updated and original damping matrices
- \([K],[K_A]\) updated and original stiffness matrices
- \([M],[M_A]\) updated and original mass matrices
- \(N\) number of updating parameters
- \(N_M\) number of rows of matrix \([A]\)
- \(N_i\) number of selected updating parameters
- \([P]\) permutation matrix
- \([Q]\) orthogonal matrix of QR decomposition
- \([R]\) upper triangular matrix
- \(r_j\) element of matrix \([R]\)
- \([U],[u]\) left singular matrix and vector
- \([V],[v]\) right singular matrix and vector
- \([X]\) vector consisting of updating parameters
- \([X^*], [X^*]^1\) selected and unselected updating parameter
- \([\Sigma]\) singular value matrix
- \(\alpha\) mass updating parameter factor
- \(\beta\) stiffness updating parameter factor
- \(\gamma\) damping updating parameter factor
- \(\varepsilon\) relative equation error norm
- \(\sigma\) singular value

**INTRODUCTION**

An accurate finite element model is always necessary for the accurate predictions of the dynamic responses of a structure and machine. To obtain such an accurate finite element model, the finite element model is usually updated by using experimental data. For an engineering structure, the locations, the types and the amounts of the erroneous elements of the model are all not known. To include all parameters of a finite element model into an updating procedure, it will absolutely result in an extraordinary large updating matrix, and the estimation of such a huge least squares problem usually gives ridiculous results. This is because the system information that can be obtained from the experimental data is very limited due to the limited measurement frequency band and the inevitable measurement noise. The resultant data matrix are usually numerically ill-conditioned and its estimation will result in sensitive solution. Practice has shown that correct localization of the finite element model errors is critical to the success of a model updating [1-4].

Many model error indicator functions have been proposed to localize the model errors. The most popular ones are the subspace searching method [11] and the QR pivoting permutation decomposition method [2]. Other methods include the residual energy method [3] and the MAC method [4]. These methods differ in the effectiveness and the efficiency, and the results are sometimes case sensitive.

The basic idea of subspace searching method comes from the concept of step-wise regression to select candidate sets of a least squares problem by Ralston and Wilf in 1967, and is used later for finite element model error localization by Lallement and Piranda [1]. It starts with one parameter and solves the reduced equation systems sequentially for each parameter. The parameter which leads to the minimal norm of all possible subsets is retained. In the second step, another parameter is added systematically and consequently from the remaining parameters to find the subset with two parameters. This procedure continues until some stop criterion is met and finally the optimal subset is found.
This method is the most popular one but the expenditure of the repeated solution of the procedure is very high, especially when the number of updating parameters is large. The flops of this procedure is positively proportional to $N_a^2N_rN_r^4$.

The dominant parameters might be found by means of the QR decomposition with column pivoting as well [6]. A QR pivoting permutation decomposition is applied to the data matrix $[A]$ to find an error indicator function to select the optimal subset of updating parameters. The procedure will result in an extraordinary large data matrix $[A]$. This method is used by Fritzen et al. [2]. Although this method is computationally efficient, it is not always effective. It has been shown that a matrix can be nearly rank deficient without a single diagonal element of matrix $[R]$ being particularly small [6]. Thus, this method by itself is not entirely reliable as a method for error localization.

In other methods, Link [3] proposes a localization method by checking the residual kinetic/potential energy for each substructure. A large residual energy value indicates an erroneous substructure, whereas a small value indicates either a small error or an insensitive substructure as in the case of a rigid substructure. In addition, the parameters corresponding to low values of residual energy may be excluded. Another method is the MAC method where a MAC or spatial-MAC sensitivity equation is used as an error indicator 141.

In this paper, an SVD/QR based model error indicator function is proposed. The theory and procedure of this method is discussed in the following section. The effectiveness and the efficiency of the proposed method is checked against those of the subspace searching method and the QR method. Test case includes a free-free beam structure. Experimental data is used to localize the erroneous parameters of the model.

**THEORY**

1. Error Localization Concept

The finite element model of a structure is usually represented by the system matrices $[M]$, $[K]$, and $[C]$. Assuming that the differences between the original finite element model and the accurate one can be expressed by a first order linear Taylor expansion, which can be true if the updating parameters are selected properly, we have

$$[M] = [M_0] + \sum \alpha_i [M]'$$

and

$$[K] = [K_0] + \sum \beta_i [K]'$$

where $\alpha, \beta$, and $\gamma$ are mostly the relative updating parameters, and $[M]', [K]'$, and $[C]'$ are the derivative matrices with respect to a specific element in the global coordinate system. The purpose of model updating is to determine the values of these coefficients by comparing the dynamic properties between the analytical and the measured data. Almost all of updating techniques lead to the estimation of a least squares problem in the form of

$$[A][X] = [B]$$

where the vector $[X]$ is a subset of all three types of updating parameters.

To include all structural parameters in the updating procedure will result in an extraordinary large data matrix $[A]$. On the other hand, the number of structural modes included in the measured data is very limited and, in addition, the model errors are mostly localized. The estimation of such a large least squares problem is usually ill-conditioned and usually produce a wrong solution. Roughly speaking, the relative estimation error is proportional to the square of the condition number of the data matrix $[A][6]$.

A commonly adopted strategy to solve this problem is to apply an error indicator function to select the optimal updating subset which consists of a reasonable number of updating parameters. The reduced equation system corresponding to this reduced subset is then better numerically conditioned, and at the same time, gives the equation residuals about each updating parameter to those from the full equation system. A reduced equation system of a least squares problem can be defined as

$$[A'][X'] = [B]$$

where $[A']$ is a subset of $[A]$ but is numerically better conditioned than $[A]$, and $[X']$ is a subset of $[X]$. The relative norm of the residuals of the reduced system is then defined as

$$\varepsilon = \frac{[B] - [A'][X']}{\| [B] \|}$$

A more stable estimation of a least squares problem can be obtained when the data matrix has a smaller condition number. The equation residuals, however, may not be improved for the reduced system. Therefore, there is a trade-off between the number of subset parameters and the minimized residuals, which is usually determined by some termination criterion.

Currently, two of the most popular subset selection methods are the subspace searching method III and the
QR permutation decomposition method [2]. In general, the subspace searching method is effective in locating the model errors but it is computationally very expensive while the QR method is efficient in computation but it is not reliable when matrix [A] is ill-conditioned. The method to be introduced in this paper is able to give a reliable error localization and, at the same time, is computationally efficient. The procedure of the new method is described in the following.

2. Singular Value Decomposition

The singular value decomposition (SVD) technique has been proven as a very effective way to determine the number of significant components in a matrix. A singular value decomposition of matrix [A] is

\[ [A] = [U][\Sigma][V]^T = [U, U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \]  

and

\[ [U]^T[U] = [I, [V][V]^T = [I] \]

or in other forms,

\[ [A] = [U, [\Sigma_1][V_1]^T + [U_2][\Sigma_2][V_2]^T \]  

and

\[ [A] = \sum_{i=1}^{N} \sigma_i [u_i][v_i]^T + \sum_{i=N+1}^{N} \sigma_i [u_i][v_i]^T \]

Assuming that matrix [A] has \( N_F \) significant singular values, the first term on the right hand of Eq. (8) contributes almost all of the useful information to matrix [A] while the second term is negligible and it mainly contributes to the measurement noise. In other words, sub-matrix [V_1] spans the subspace of the useful data information, and sub-matrix [V_2] spans the subspace of the unwanted information, of matrix [A]. Therefore, the second term in Eq. (8) can be discarded.

3. QR Pivoting Permutation Decomposition

By QR pivoting permutation decomposition, a matrix can be decomposed into the product of an orthogonal, or the base, matrix [Q] and an upper triangular matrix [R],

\[ [A]_{\text{perm}} = [A][P] = [Q][R] \]

where [P] describes the interchange of the columns of matrix [A]. The diagonal elements of matrix [R] are arranged in non-ascending order of

\[ r_{11} \geq r_{22} \geq \cdots \geq r_{nn} \]

The \( j \)th column of the permuted matrix \([A]_{\text{perm}}\) is

\[ \{A_{\text{perm}}\}_j = r_1 [Q]_1 + r_2 [Q]_2 + \cdots + r_j [Q]_j, j = 1, 2, \ldots, N \]

The \( j \)th column of the permuted matrix \([A]_{\text{perm}}\) is a linear combination of the first \( j \) column vectors of the orthogonal matrix [Q], and the \( j \)th column vectors of the \( j \)th column of matrix [R] are the coefficients. In most cases, the diagonal terms of matrix [R] are dominant compared to the off-diagonal terms. If starting from a specific value \( j = N_F + I \), \( r_{ij} \) begins to become relatively small, all further column vectors \([A]_{\text{perm}}\) are approximately the linear combination of the first \( N_F \) permuted column vectors. Then, it is concluded that QR with column pivoting “discover” the rank \( N_F \) of the matrix [A]. Unfortunately, it is not always effective. It has been shown that a matrix can be nearly rank deficient without a single \( r_{ij} \) being particularly small [6]. And it is not always that the upper off-diagonal terms of matrix [R] are small enough. Thus, this method by itself is not entirely reliable as a method for detecting near rank deficiency.

4. SVD/QR Based Model Error Indicator Function

The right singular matrix \([V]\) of the data matrix \([A]\) has very important physical significance. Its rows mark the positions of the updating parameters. Its column vectors indicate the contributions to the matrix [A] in such a way that the first column contributes the most and the last column contributes the least. For an ill-conditioned problem, the contributions from the last few columns can be neglected without introducing large equation residual norm errors. Therefore, if the data matrix \([A]\) is ill-conditioned and has a rank of \( N_F \) where \( N_F < N \), a singular value decomposition can be used to determine the acceptable rank for the data matrix [A]. For the right singular matrix \([V]\), retain the useful part \([V_1]\) and discard \([V_2]\) which mostly contributes to measurement noise. Then, a QR decomposition with column pivoting is applied to matrix \([V_1]\) to find the important columns \([A]\) of \([A]\). This procedure is defined as the Singular Value Decomposition with QR pivoting permutation decomposition (SVD/QR) based model error indicator function and is summarized below,

1) Conduct the singular value decomposition on the data matrix \([A]\) as described in Eq. (7).
2) From the singular value matrix \([\Sigma]\), determine the rank \( N_F \) of matrix [A].
3) Apply QR decomposition with column pivoting to the sub-matrix \([V_1]\) to find the permutation matrix \([P]\),

\[ [Q]^T[V_1]^T[P] = [R_1, R_2] \]

4) Use permutation matrix \([P]\) to find the reduced data matrix \([A']\) and the subset \([X']\),

\[ [A'] = [A']_{\text{perm}} [P] \]

574
The flops of this procedure are proportional to $N_m \times N_p^2$ only. Comparing to the flops of $N_m^2 \times N_p^4$ required by the subspace searching method [1], this is a huge saving of computational cost. Since the norm of the difference of the residuals between the original and the reduced systems is only dependent on the conditions of the sub-matrix $[V_1]$ and since the part that is dominated by measurement noise and has little contributions is discarded, the new error indicator function can expect better localization.

**NUMERICAL EXAMPLE**

To verify the presented SVD/QR error indicator function, a hot-rolled steel plate of $0.0762 \times 0.00635 \times 1.2192$ m$^3$ ($3'' \times 
\frac{1}{64}'' \times 48''$) with an elastic modulus of 210 GPA and a mass density of 7.87 kg/m$^3$ was used. The steel plate is considered as a two dimensional free-free beam. An impact test strategy was then applied to collect the test data. In total, 25 impacting points were evenly selected along the length of the beam with respect to one reference at one end (point 1) of the beam. Only transverse motions were taken into consideration. Therefore, the test model has 25 degrees of freedom.

In the test, a frequency resolution of 1 Hz was chosen in the range from 0 to 400 Hz, and all measurements were averaged five times. In total, 25 frequency response functions (FRFs) were collected. Figures 1a) and 1b) show the measured frequency response functions at driving point 1 and at cross point 14 respectively.

The beam is then analytically modeled using a finite element method. In total, 48 two-dimensional beam finite elements are used which form an analytical model of 98 degrees of freedom (see Figure 2). A reduction procedure [8] is applied to match the number of degrees of freedom between the test and the analytical models. In constructing the finite element model, the stiffness at the middle part of the beam was artificially modeled using half of the nominal value. This modeling error causes the frequency differences between the analytical finite element model and the measured structure to range from -15% to -30% for all measured modes. Table 1 compares the analytical and the measured modal frequencies.

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>FE Model</th>
<th>Measured</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16.480</td>
<td>22.748</td>
<td>-27.55</td>
</tr>
<tr>
<td>2</td>
<td>26.622</td>
<td>52.030</td>
<td>-21.62</td>
</tr>
<tr>
<td>3</td>
<td>102.61</td>
<td>122.57</td>
<td>-16.28</td>
</tr>
<tr>
<td>4</td>
<td>174.13</td>
<td>203.77</td>
<td>-14.54</td>
</tr>
<tr>
<td>5</td>
<td>254.85</td>
<td>304.29</td>
<td>-16.25</td>
</tr>
</tbody>
</table>

In localizing the potential model errors, a least squares problem with respect to all potential updating parameters is formed using the updating procedure described in reference [7], where a response residual criterion is used and the experimental data is the frequency response functions. For each element, there are two updating parameters -- the mass density per unit length and the bending stiffness -- which result in 96 element parameters in total. Since this number is too large when compared to the number of modes, a better strategy is to group these parameters first and then locate the potential erroneous groups. In this case, 13 parametric groups are constructed and they are illustrated in Figure 2. Since the total mass of the beam is easily found from the weight and each element is the same size, the mass group is not included in the localization.

Eight updating frequencies are selected in such a way that two frequencies are taken around each measured mode frequencies in the frequency range from 50 to 310
Hz. Frequencies around the first mode are not selected because the data in this range has very low coherence values. All 25 measured frequency response functions are used. As a result, the size of the final data matrix $[A]$ is 200x12.

The first step in localizing the model errors is to check the number of significant singular values of the data matrix $[A]$. Figure 3 plots the singular values, and it indicates the number of significant singular values is 10. This number defines the number of columns in the sub-matrix $[V_1]$ that should be used in determining the permutation matrix $[P]$ by Eq. (12).

![Singular Values of Data Matrix](image)

**Figure 3. Data Matrix Singular Values**

Although all updating groups may contain errors, the dominant errors should come from stiffness groups 4, 5, 6, 7, 8, and 9 because of the obviously wrong assumption on stiffness for those groups. A correct localization by an error indicator function should be able to detect these six groups.

The effectiveness of the presented error indicator function is checked against two most popular error indicator functions — the subspace searching method and the QR decomposition method. Figure 4 plots the relative equation residual error norms, from which the proper order of the sub-system can be determined by looking at the cut-off point. Thus, the correct order by the SVD/QR method, when the significant number of singular value is set at 10, is six, and that by the subspace searching method is seven. The corresponding parametric groups so selected are listed in Table 2. The numbers that are above the double line are the selected ones, and those in bold correspond to the stiffness parameters of the middle portion of the test beam. It is clear that these two methods locate the dominant model errors very well. On the contrary, the QR method, and the SVD/QR method when the significant number of singular value is wrongly set as at 12, do not correctly locate the dominant model errors.

![Relative Equation Error Norms](image)

**Figure 4. Relative Equation Error Norms**

<table>
<thead>
<tr>
<th>NO.</th>
<th>SVD/QR Method (svd=10)</th>
<th>Subspace Method</th>
<th>QR Method</th>
<th>SVD/QR Method (svd=12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>8</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>4</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>10</td>
<td>2</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>7</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>9</td>
<td>10</td>
<td>9</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>6</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>2</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>3</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>11</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>1</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>12</td>
<td>12</td>
<td>1</td>
<td>10</td>
</tr>
</tbody>
</table>

The localized first seven erroneous elements, that are elements 4, 5, 6, 7, 8, 9, and 10, are then used as the updating parameters to update the finite element model of the beam. The dynamic properties of the updated model are then calculated. It is found that the maximum
error among the first five mode frequencies in the range from 0 to 400 Hz between the updated model and the measured structure is only 0.84%, down from the original 27.6%. Figure 6 compares the frequency response functions at cross point 14 among the measured, the one from the original finite element model and the one from the updated model. It is obvious that excellent matches exist across the whole frequency range.

![Figure 5. Condition Numbers of Subsystems](image)

The efficiency of a model error indicator function should also be noticed. The computational cost is completely dependent on the size of data matrix $[A]$ where the number of columns is the number of updating groups, and the number of rows is the multiplication of the numbers of measured responses, references, updated frequencies, and if any, structural configurations. The computational efficiency of the presented SVDIQR method is checked under MATLAB 4.2c environment on a HP-9000 J200 (770) workstation and compared with the subspace searching method and the QR decomposition method. The first case is to find the CPU time as a function of the number of updating frequencies where the number of updating parametric groups is fixed at 13 and the number of updating frequencies varies. Since all 25 responses are used, the size of the data matrix $[A]$ becomes $(25 \times$ the number of updating frequencies) by 13. Listed in Tables 3 is the CPU time in seconds when the number of updating frequencies are set at 6, 18, 30, 42, and 54 respectively. The second case is to find the CPU time as a function of the number of updating parametric groups where the number of updating frequencies is fixed at 30 while the number of updating parametric groups varies. Hence, the size of the data matrix $[A]$ becomes $(25 \times 30)$ by (the number of updating groups). Table 4 is the CPU time in seconds when the number of updating parametric groups are set at 5, 9, 13, and 25 respectively. For better visualization, the CPU time in the two cases are plotted in Figures 7 and 8 respectively.

From Tables 3 and 4 and Figures 7 and 8, the CPU time required by the SVD/QR Method and the QR decomposition method is shown to be small and increases slowly with the increase of the number of updating frequencies and the number of updating groups. This total CPU time is quite reasonable for a practical model error localization. The CPU time required by the subspace searching method for the same problems is, however, larger and increases cubically with the increase of the number of updating frequencies and quadruply with the increase of the number of updating groups. This computational cost is very intensive and become generally unacceptable in updating the models of an engineering structures.

![Figure 6. FRF Comparison](image)

<table>
<thead>
<tr>
<th>Table 3. CPU Time vs. Updating Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No. of Frequencies</strong></td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>18</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>42</td>
</tr>
<tr>
<td>54</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 4. CPU Time vs. Updating Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No. of Groups</strong></td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>13</td>
</tr>
<tr>
<td>25</td>
</tr>
</tbody>
</table>
SUMMARY

This paper presents a new model error indicator function, the \textit{SVD/QR Based Model Error Indicator Function}. First, it uses singular value decomposition technique to select the effective rank of the data matrix and the corresponding sub-right singular matrix that is less noise contaminated and numerically better conditioned. Then, a QR permutation decomposition with column pivoting is applied to this sub-matrix to find the permuted parametric groups. Finally, the relative equation error norms corresponding to sequentially selected reduced systems are calculated and an error termination criterion is applied to determine the number of parametric groups that contain potential modeling errors.

The presented method is verified through the application to a free-free test beam, and the results are compared with the most popular \textit{subspace} searching method and the QR permutation decomposition method. In general, the popular \textit{subspace} searching method is effective in locating the model errors but it is computationally very extensive. The QR decomposition method, though it costs much less in computation, does not work effectively for the problem under investigation. However, the presented \textit{SVD/QR} based method works as effectively as the \textit{subspace} searching method and is as efficient as the QR decomposition method.

REFERENCES


