Nonlinear Modal Interactions of a Cantilever Beam with an Internal Resonance

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ABSTRACT. Nonlinear torsional-flexural modal interactions of a torsionally excited cantilever beam are studied in the case of one-to-one internal resonance. Discrete two DOF model is derived from the equations of motion for the continuous cantilever beam by using the Galerkin mode approximation method. The method of averaging is used to obtain the approximation to the response of the discrete model. Steady state constant and periodic solutions are studied by analytical and numerical approach. A bifurcation analysis of the averaged equations is undertaken, and it is shown that torsional-flexural coupled response arises when the torsional mode becomes unstable for some excitation frequencies and amplitudes, the coupled response results from energy transfer between the two nonlinearly coupled modes. The coupled response can lead to amplitude-and-phase-modulated response via Hopf bifurcation.

2. EQUATIONS OF MOTION

The system analyzed here is shown in Fig.1 In order to perform a nonlinear analysis effectively, it was introduced the following assumptions:

(i) A cantilever beam has thin and uniform cross-section with its length (i.e., like "elastica"[1, 2] shape) and its physical properties (e.g., Young's modulus, shear modulus, mass density etc.) are invariant and homogeneous.
(ii) A cantilever beam is satisfied the inextensibility and the unshearability conditions.

Upon the assumptions, it is subjected to a general "torsional" excitation $q(t)\cos(\Omega t)$. However, the partial differential equations, and boundary conditions, that govern the three dimensional flexural torsional motions for inextensional beams, subjected to a distributed periodic force, are formulated in [3], and further details of the formulation are also given in [4]. And for convenience, after introducing the nondimensional quantities and by making use of the angle of torsion for the beam[5], the normalized form of equations may be written as given below for the case when the beam is subjected to the torsional excitation indicated above.

$$
\ddot{v} + \gamma \dot{v} + \beta_1 \ddot{w}'''
\left[ -\beta_2 \int_0^l (v''')^2 + \int_0^l (v''')^2 v'''' ds - \int_0^l (v''')^2 v''' ds 
+ \frac{1}{2} \int_0^l (v''')^2 \, ds 
+ j \int_0^l (v''')^2 \, ds 
+ \frac{1}{2} \int_0^l (v''')^2 \, ds 
+ j \int_0^l (v''')^2 \, ds 
\right]
$$

1. INTRODUCTION

In this work, the primary resonant nonlinear response of a cantilever beam in the presence of a one-to-one internal resonance is studied. Linear formulation of beam problems show that the motions in the two principal planes are independent; consequently, forced motion in one principal plane is always stable in that plane. But when the vibration amplitude is large, nonlinear effects become significant, and several nonlinearities come into play.

So, many researchers have studied the nonlinear coupled response analysis of a cantilever beam under various excitations for some practical use, e.g., helicopter rotor blades, manipulator arms, antennas, and so on [1-7]. However, they focused the out-of-plane torsion-like motions under in-plane lateral and / or axial excitation. In this study, we investigate to make an approximated nonlinear system, which can be easily examined, and to analyze the bending-torsion coupled motions under torsional base excitation.
In the above equations, \( I, y, M' \) denote \( \gamma \)-directional, \( z \)-directional and \( \varphi \)-rotational angular displacement, respectively. The principal axes of the beam's cross section at location \( s, y, z \) are shown in Fig. 1. The quantities \( \beta \), \( \alpha \), \( \gamma \) are the ratios of the principal bending and torsional stiffnesses to the principal \( \gamma \)-directional bending stiffness: \( j_i \)'s (\( i = \xi, \eta, \zeta \)) are the constant distributed mass moment of inertia.  \( \left( \right) \) and \( \left( \right)^* \) denote, respectively, partial differentiation with respect to \( s \) and \( t \).

The solution to the undamped linearized differential equations of motion will be used as the starting point for a nonlinear analysis based on Eq.(1a-c). Those solutions are of the form

\[
\begin{align*}
\gamma(s,t) &= F_\gamma(s)\gamma(t) \\
n(s,t) &= F_n(s)n(t) \\
\gamma(s,t) &= F_\gamma(s)\gamma(t)
\end{align*}
\]

In general, the eigenfunctions \( F_\gamma(s), F_n(s) \) and \( F_\gamma(s) \) are determined by solving the linearized counterpart to Eq.(1a-c); the time-variant functions of the solutions are described by harmonic functions with appropriate linear natural frequencies and phases. Then each eigenfunction represents each mode of vibration as being investigated. Applying Galerkin's mode \textit{approximation} method [5], which uses the mode \textit{orthogonality} and the boundary conditions of the beam, it can be obtained the ordinary differential equations with respect to \( s \) for three mode approximated model.

\[
\begin{align*}
\gamma_t + c_n \gamma + \omega_n^2 \gamma &= \alpha_n \gamma, w_i + \alpha_n \gamma, w_i, + \alpha_n \gamma, w_i, + \gamma, \gamma, \gamma, \gamma, \\
\gamma_t + c_n \gamma + \omega_n^2 \gamma &= \alpha_n \gamma, w_i + \alpha_n \gamma, w_i, + \alpha_n \gamma, w_i, + \gamma, \gamma, \gamma, \gamma, \\
\gamma_t + c_n \gamma + \omega_n^2 \gamma &= \alpha_n \gamma, w_i + \alpha_n \gamma, w_i, + \alpha_n \gamma, w_i, + \gamma, \gamma, \gamma, \gamma,
\end{align*}
\]

In the above equations, \( \gamma, w_i, \gamma \) denote \( \gamma \)-directional, \( z \)-directional and \( \varphi \)-rotational angular displacement, respectively. The principal axes of the beam's cross section at location \( s, \xi, \eta, \zeta \) are shown in Fig. 1. The quantities \( \beta \), \( \alpha \), \( \gamma \) are the ratios of the principal bending and torsional stiffnesses to the principal \( \gamma \)-directional bending stiffness: \( j_i \)'s (\( i = \xi, \eta, \zeta \)) are the constant distributed mass moment of inertia.  \( \left( \right) \) and \( \left( \right)^* \) denote, respectively, partial differentiation with respect to \( s \) and \( t \).

The solution to the undamped linearized differential equations of motion will be used as the starting point for a nonlinear analysis based on Eq.(1a-c). Those solutions are of the form

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\gamma(s,t) &= F_\gamma(s)\gamma(t) \\
n(s,t) &= F_n(s)n(t) \\
\gamma(s,t) &= F_\gamma(s)\gamma(t)
\end{align*}
\]
The numerical values of the coefficients associated with the mass moment of inertia were found to be negligible when compared to the values for the coefficients of the similar terms associated with the stiffness. For these reasons, the $j_i, j_2$ terms will be neglected, thus simplifying analysis. However, when the torsional natural frequency $\omega_\tau$ is commensurable with the bending natural frequencies, the term $j_1 j_2$ in Eq.(1c) is associated with the linearized counterpart of that equation and, thus cannot be neglected. In such a case, it can be found the energy exchange between torsional and bending motions, which results from the internal resonance.

Simplifying as discussed above, the following equations of motion are obtained:

$$\ddot{v} + j \omega_v^2 v = \alpha_{v1} y_1 v + \alpha_{v2} v_1 y_1 + \alpha_{v3} v_1 v_1 + \alpha_{v4} v_1^2$$

$$\ddot{w} + j \omega_w^2 w = \alpha_{w1} y_1 w + \alpha_{w2} w_1 y_1 + \alpha_{w3} w_1 w_1 + \alpha_{w4} w_1^2$$

$$\ddot{y} + j \omega_y^2 y = \alpha_{y1} v_1 y + \alpha_{y2} v_1 v_1 + \alpha_{y3} v_1 v_1 + \alpha_{y4} v_1^2$$

(4a)

In order to analyze the motion governed by the coupled nonlinear Eq.(4a-c), the method of averaging will be used. We introduce a small, “book-keeping” parameter $\varepsilon$, such that

$$v = \varepsilon v, w = \varepsilon w, y = \varepsilon y$$

(5a)

By transferring the damping and the excitation terms out of the $O(\varepsilon)$ approximation as

$$c_1 = \varepsilon^3 c_1, c_2 = \varepsilon^3 c_2, c_3 = \varepsilon^3 c_3$$

$$f_{v1} = \varepsilon f_{v1}, f_{w1} = \varepsilon f_{w1}$$

(5b)

Eq.(6a-c) are obtained for each order of approximation when the expansions defined by Eq.(5a-c) are substituted into Eq.(4a-c):

$$\ddot{v} + \varepsilon^2 \omega_v^2 v = \varepsilon^2 A_1 v + \varepsilon^2 A_2 v_1 y + \varepsilon^2 A_3 v_1 v + \varepsilon^2 A_4 v_1^2$$

$$\ddot{w} + \varepsilon^2 \omega_w^2 w = \varepsilon^2 B_1 w + \varepsilon^2 B_2 w_1 y + \varepsilon^2 B_3 w_1 w + \varepsilon^2 B_4 w_1^2$$

$$\ddot{y} + \varepsilon^2 \omega_y^2 y = \varepsilon^2 C_1 y + \varepsilon^2 C_2 y_1 v + \varepsilon^2 C_3 y_1 v_1 + \varepsilon^2 C_4 y_1^2$$

(6a)

The newly introduced coefficients $A's, B's, C's$ are listed in Appendix, (A.2). Then it will be analyzed with quadratic and cubic nonlinear terms (i.e., $O(\varepsilon^3)$ terms are neglected). And new symbols in Eq.(7) have been introduced for brevity of the expressions.

$$x_1 = v, \quad x_2 = y$$

$$x_3 = w, \quad x_4 = y$$

(7)

Then the coupled second-order differential equations (6a-c) can be expressed in vector form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\omega_v^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\omega_w^2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} + \varepsilon f_1 + \varepsilon^2 f_2$$

(8)

where

$$f_1 = \begin{bmatrix} 0 \\ 0 \\ -B_2 y_1 z_1 \\ 0 \end{bmatrix}$$

$$f_2 = \begin{bmatrix} 0 \\ -C_1 x_1 y_1 - C_2 x_2 y_2 \\ 0 \\ 0 \end{bmatrix}$$

$$r_1 = \begin{bmatrix} 0 \\ 0 \\ -A_1 x_1 v_1^2 + A_2 x_1 y_1^2 + A_3 x_1 v_1^2 + A_4 x_1 v_1^2 + c_v v_1^2 \\ 0 \\ -B_1 y_1 v_1^2 + B_2 y_1 y_1^2 + B_3 y_1 v_1^2 + B_4 y_1 v_1^2 + c_v y_1^2 \\ 0 \\ -C_1 y_1^2 + C_2 y_1^2 + c_v - f_w \cos(\Omega t) \end{bmatrix}$$
To apply the method of averaging, Eq. (8) needs to be transformed into the "standard form" [8]. For this, we let

$$\mathbf{x}(t) = e^{t\mathbf{X}} \mathbf{X} = \mathbf{x}(t), \quad (9)$$

Substituting Eq. (9) into Eq. (8) gives

$$\dot{\mathbf{x}} = \varepsilon \mathbf{g}_1 + \varepsilon^2 \mathbf{g}_2$$

Thus Eq. (14a-b) are also transformed into the standard form of Eq. (10). It will be averaged with respect to the minimum period with the Krylov-Bogoliubov sense [8]. The averaged equations corresponding to the original system (10) are given by

$$\mathbf{X}' = \varepsilon \mathbf{g}_1(\mathbf{X}) + \varepsilon^2 \mathbf{g}_2(\mathbf{X}) \quad , \quad (15)$$

where $\mathbf{X}$ represents the averaged vector of Eq. (9). In the present work, however, the first-order terms are out of scope as shown in Eq. (14a-b). Thus Eq. (15) represents the averaged equations of the discrete two DOF model. It can be written in first-order four differential equations such as

$$X'_1 = \varepsilon^2 \left[ -\frac{1}{2} c_1 X_1 + \frac{1}{2} (\sigma_1 + \sigma_2) X_1 + a_1 X_1^2 + a_2 X_1^3 \right]$$

$$X'_2 = \varepsilon^2 \left[ -\frac{1}{2} c_2 X_2 + \frac{1}{2} (\sigma_1 + \sigma_2) X_2 + a_3 (X_2 Z_1 + 2 X_2 Z_2 + 3 X_2 Z_2^2) \right]$$

$$Z'_1 = \varepsilon^2 \left[ -\frac{1}{2} c_1 Z_1 + \frac{1}{2} \sigma_1 Z_1 \right]$$

$$Z'_2 = \varepsilon^2 \left[ -\frac{1}{2} c_2 Z_2 + \frac{1}{2} \sigma_2 Z_2 \right]$$

In Eq. (16a-d), $X_1$, $X_2$, and $Z_1$, $Z_2$ are associated with the bending mode and the torsional mode of the autonomous system, respectively. The coefficients $a_1$, $a_2$ and $a_3$ are defined in Appendix, (A.3).

3. STEADY-STATE SOLUTIONS AND BIFURCATION ANALYSIS

Analysis of steady state constant solutions is accomplished when the averaged equations (16a-d) are transformed to the following polar coordinates:

$$x_i = R_i \cos(\phi_i)$$
\[ X_1 = R_1 \sin(\phi_1), \]
\[ Z_1 = R_1 \cos(\phi_1), \]
\[ Z_2 = R_1 \sin(\phi_1). \]

The resulting equations in polar form are

\[ R_i' = -\frac{1}{2} c_i R_i - a_i R_i^2 \sin(2\phi_i - 2\phi_1), \]  
\[ R_i \phi_i' = \frac{1}{2} (\sigma_i + \sigma_2) R_i - a_i R_i^2 \]
\[ -a_i R_i R_i^3 \left[ 2 + \cos(2\phi_i - 2\phi_1) \right], \]  
\[ R_i' = -\frac{1}{2} c_i R_i + \alpha_i R_i^2 R_i \sin(2\phi_i - 2\phi_1) + \frac{1}{2} f_{\phi_i} \sin(\phi_i), \]  
\[ R_i \phi_i' = \frac{1}{2} (\sigma_i + \sigma_2) R_i - \alpha_i R_i^2 R_i \left[ 2 + \cos(2\phi_i - 2\phi_1) \right] + \frac{1}{2} f_{\phi_i} \cos(\phi_i). \]

From the averaged Eq.(16a-d), the single-mode steady state constant solutions, which correspond to the uncoupled torsional motion, can be obtained as

\[ R_i = \frac{f_{\phi_i}}{\sqrt{\sigma_i^2 + C_i^2}}, \quad \tan \phi_i = -\frac{C_i}{\sigma_i}. \]  

The steady-state constant solutions for which both \( R_i \neq 0 \) and \( R_i' \neq 0 \), called the coupled-mode solutions, are obtained from the cubic equations for the amplitude-related quantities \( R_i^2 \) and \( \phi_i \). Solving these cubic equations analytically, however, is very tremendous. We can obtain the numerical solutions for the original averaged Eq.(16a-d) instead of the amplitude-phase Eq.(16a-d) by using well-known analysis program named “AUTO” [9].

The steady state response plots are shown in Fig.2 to Fig.6. For an internal resonance condition, it can be shown that the steady state coupled-mode responses are bifurcated from the steady state single-mode (i.e., torsional mode) when the single-mode solution becomes unstable in the state of appropriate parameters. The solid lines in the figures represent the stable modes; while the dotted lines represent the unstable modes of vibration. Especially in Fig.5 and Fig.6, it can be found the “pitchfork” bifurcation points and the “Hopf” bifurcation points. And we can also find the stable and unstable periodic solutions which are bifurcated from the coupled-mode constant solution using AUTO. The various bifurcation analyses, which are both analytically and numerically, will be the subjects of the future study.

The eigenvalues of the Jacobian matrix for the averaged system (16a-d), which determine the stability of the single-mode steady state solution, can be shown to satisfy the two quadratics

\[ \lambda^2 + c_i \lambda + \frac{1}{4} (\sigma_i + \sigma_2)^2 \]
\[ -2 \alpha_i (\sigma_i + \sigma_2) (Z_i^2 + Z_i^3) + 3 \alpha_i^2 (Z_i^2 + Z_i^3)^2 = 0 \]  
and

\[ \lambda^2 + c_i \lambda + \frac{1}{4} \sigma_i^2 = 0. \]

From Eq.(20a-b), it is seen that no eigenvalue can be pure imaginary for nonzero dampings and, as a result, Hopf bifurcation cannot arise from the single-mode steady state solutions. Therefore, the single-mode steady state solutions can lose stability only when an eigenvalue become zero. Using Eq.(19), Eq.(20a) and Eq.(17), the condition for loss of stability of the single-mode steady state solutions can be shown to be

\[ \frac{1}{4} c_i^2 + \frac{1}{4} (\sigma_i + \sigma_2)^2 - 2 \alpha_i (\sigma_i + \sigma_2) R_i^2 + 3 \alpha_i^2 R_i^4 = 0 \]  
and

\[ R_i^2 = \frac{1}{\pi^2} \left[ \frac{1}{\sigma_i^2} \left( \frac{c_i^2}{c_i^2 + \sigma_i^2} - 1 \right)^2 + \left( \frac{c_i}{c_i^2 + \sigma_i^2} \right)^2 \right]. \]

Then the pitchfork bifurcation set is obtained from the above equations, and Fig.7 shows it in the \((\sigma_i, \sigma_2)\) plane. Thus, the coupled-mode steady state response arises by a pitchfork bifurcation where the single-mode steady state solution undergoes a change in stability.

4. CONCLUSIONS

From the study for the nonlinear vibration analysis of a cantilever beam with an internal resonance, we can conclude as:

(i) Discrete two DOF (bending-torsion) model is derived from the equations of motion for the torsionally excited continuous cantilever beam by using the Galerkin’s mode approximation. The method of averaging is used to obtain the approximation to the response of the discrete model in the case of one-to-one internal resonance.

(ii) Steady state constant and periodic solutions are studied by analytical and numerical approach. A bifurcation analysis of the averaged equations is performed, and it is shown that torsional-flexural coupled response arises when the torsional mode becomes unstable for some excitation frequencies and amplitudes. The coupled response results from “energy transfer” between the two nonlinearly coupled modes.
(iii) The steady state periodic solutions bifurcate from the coupled-mode steady state constant solutions via Hopf bifurcation for some parameters. The coupled-mode steady state response arises by a pitchfork bifurcation where the single-mode steady state solution undergoes a change in stability. In addition, the pitchfork bifurcation set is obtained analytically.

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APPENDIX

(A.1)

\[ \alpha_{i,n} = -\beta_{i,1} \int_{0}^{1} F_i \left( \frac{d}{ds}(F_i)' \right)' ds + (1 - \beta_{i,1}) F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,2} = -(1 - \beta_{i,1}) \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,3} = \int_{0}^{1} \left( (1 - \beta_{i,1}) F_i \left( F_i'' \right)'' + \frac{1}{2} \left( F_i'' \right)' \right) ds \]

\[ \alpha_{i,4} = j_{i,4} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,5} = -j_{i,5} c_{i,5} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,6} = \int_{0}^{1} \left[ (1 - \beta_{i,1}) F_i \left( F_i'' \right)' + \left( 1 - \beta_{i,1} \right) F_i \left( F_i'' \right)' \right] ds \]

\[ \alpha_{i,7} = -\beta_{i,1} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,8} = -\beta_{i,1} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,9} = j_{i,9} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,10} = -j_{i,10} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,11} = j_{i,11} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,12} = -j_{i,12} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,13} = j_{i,13} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,14} = -j_{i,14} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,15} = j_{i,15} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,16} = -j_{i,16} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,17} = j_{i,17} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,18} = -j_{i,18} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,19} = j_{i,19} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,20} = -j_{i,20} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,21} = j_{i,21} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,22} = -j_{i,22} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,23} = j_{i,23} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,24} = -j_{i,24} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,25} = j_{i,25} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,26} = -j_{i,26} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,27} = j_{i,27} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,28} = -j_{i,28} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,29} = j_{i,29} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,30} = -j_{i,30} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,31} = j_{i,31} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,32} = -j_{i,32} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,33} = j_{i,33} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,34} = -j_{i,34} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,35} = j_{i,35} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,36} = -j_{i,36} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,37} = j_{i,37} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]

\[ \alpha_{i,38} = -j_{i,38} \int_{0}^{1} F_i \left( F_i'' \right)' ds \]
\[ \alpha_{m} = \frac{1}{8} \left( \omega_{m}^{3} A_{1} + A_{c} \right), \quad \alpha_{2} = \frac{1}{8 \omega_{c}^{3}} A_{2}, \quad \alpha_{i} = \frac{1}{8 \omega_{i}^{3}} C_{i} \]

(A.3)

REFERENCES


Fig.1 Configuration of a cantilever beam
Fig. 2 Amplitude response plot of $R_1^2 + R_2^2$ in case of $\sigma_\gamma = 0.0, \epsilon_1 = \epsilon_\gamma = 0.02, F_\gamma = 0.005$

Fig. 3 Amplitude response plot of $R_1^2 + R_2^2$ in case of $\sigma_\gamma = 0.1, \epsilon_1 = \epsilon_\gamma = 0.01, F_\gamma = 0.01$

Fig. 4 Amplitude response plot of $R_1^2 + R_2^2$ in case of $\sigma_\gamma = 0.1, \epsilon_1 = \epsilon_\gamma = 0.01, F_\gamma = 0.01$

(Detailed plot of Fig. 3)

Fig. 5 Hopf bifurcation points and responses in case of $\sigma_\gamma = 0.1, \epsilon_1 = \epsilon_\gamma = 0.01, F_\gamma = 0.01$

(Detailed plot of Fig. 4)

Fig. 6 Coupled bending amplitude response plot of $X_1$ in case of $\sigma_\gamma = 0.1, \epsilon_1 = \epsilon_\gamma = 0.01, F_\gamma = 0.01$

Fig. 7 Pitchfork bifurcation set in case of $\epsilon_1 = \epsilon_\gamma = 0.01, F_\gamma = 0.01$