A SOLUTION METHOD FOR SOME NONLINEAR EIGEN-PROBLEMS IN STRUCTURAL DYNAMICS

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ABSTRACT
Frequency-dependent matrices in structural dynamics result in a non-linear eigenvalue problem. The advantage of frequency-dependent matrices is that the order is usually smaller for comparable accuracy of the eigenparameters of a given system. This paper presents a method for solving non-linear eigenvalue problems, in which the matrix function of the eigenvalue can be expanded in terms of its powers. The solution approach consists of first solving the basic linear eigenvalue problem and then using the higher order matrices to converge the eigenvalues one at a time. A perturbation approach is adopted to derive the convergence scheme in which separate iterations are performed for the eigenvalue and eigenvector during each step. This method is an alternative to the classical companion matrix method which can only be used for problems that have weak nonlinearity, i.e., which have series expansions up to a small power of the eigenvalue. Also, a method is introduced to perform dynamic condensation for finding approximate eigenvalues when only a small fraction of the total eigenvalues are wanted. Numerical examples demonstrate the application of the method.

Introduction
The use of frequency-dependent matrices in structural dynamics leads to eigenvalue problems that are non-linear in the eigenvalue $\lambda$. This is in contrast to the classical linear eigenvalue problem which results when frequency-independent matrices are used. In general, the order of frequency-dependent matrices is smaller than that of frequency-independent matrices for comparable accuracy in the eigenparameters. In what follows, eigenvalue problems that can be described by the equation

$$\bar{N}(\lambda) \bar{q} = 0$$

are considered, where $\bar{N}(\lambda)$ is a symmetric matrix and is a function of $\lambda$, and is such that it can be expanded in series form as

$$\bar{N}(\lambda) = \bar{N}_0 + \lambda \bar{N}_1 + \lambda^2 \bar{N}_2 + \ldots + \lambda^r \bar{N}_r,$$

where, $r$ may be either finite or infinite, and the matrices $N_s, s = 0, 1, 2, \ldots, r$, are constant coefficient matrices. It is also assumed that the system is positive semi-definite and so is the basic linear system, i.e.,

$$[\bar{N}_0 + \lambda \bar{N}_1] \bar{q} = 0.$$  

Several methods exist in the literature for solving the non-linear eigenvalue problem described by equation (1). However, these methods involve expensive factorizations, forward and backward substitutions or a truncation of the infinite series up to a small finite
power of $\lambda$, along with the formation of block-companion matrices.

Some of the algorithms in the literature use the vanishing determinant property of the $h$-matrix at the exact eigenvalues. Kublanovskaya\cite{1} uses a unitary transformation to obtain a scalar equation in terms of the eigenvalue without evaluating the determinant explicitly, followed by the application of Newton Raphson iteration for root finding. Leung's algorithm\cite{2} is also based on a similar approach, with the eigenvectors obtained by inverse iteration. An algorithm by Thurston\cite{3} is based on linearization of the non-linear eigenvalue about an approximate root and solving many linear eigenvalue problems till convergence is obtained. Each eigenvalue is converged separately. Ruhe\cite{4} studied algorithms which were extensions of those used for linear eigenvalue problems, such as inverse iteration and the QR algorithm. He also considered an algorithm that reduces the non-linear eigenvalue problem into a series of linear problems. Rajakumar\cite{5} uses a Lanczos two-sided recursion algorithm with bi-orthogonal transformations for quadratic eigenvalue problems.

Moler and Stewart\cite{6}, and Fricker\cite{7} have developed algorithms based on block companion matrices. These matrices are obtained using the coefficient matrices of the non-linear matrix eigenvalue problem and essentially reduce the non-linear problem to a linear problem of larger order. Moler and Stewart\cite{6} solved the resulting linear problem using a QZ algorithm. Fricker\cite{7} defined master and slave coordinates for a reduction transformation based on a cut-off frequency. The resulting problem is cast in a block-companion matrix form for solution using a linear algorithm. Jain, Singhal and Huseyin\cite{8} developed an algorithm for finding only the eigenvalues through a triangularization scheme for calculating the determinant. Yang\cite{9} also used a similar factorization for evaluating the determinant. Both these methods were able to calculate the eigenvalues only. Osborne and Michaelson\cite{10} developed a method of iteration similar to the method to be developed here, but their method involves coupling between the eigenvalue and eigenvector. This requires factorizations and forward and backward substitutions at each stage of the iteration.

Methods involving the formation of block companion matrices are not useful if exact solutions to the infinite-series eigenvalue problem are desired. Additionally, these methods are cumbersome to implement for large values of $r$ in equation (2). Truncation to a small power is an approximation used in many methods, but they will not provide exact solutions to equation (1). Many methods are only able to solve for the eigenvalues. Methods involving factorization and forward and backward substitutions are expensive to implement and inherit all the numerical difficulties associated with the factorization of matrices, such as ill-conditioning and near-zero pivots.

Here, the objective is to develop a simple method which takes advantage of the special structure of the matrices $N_0$ and $N_1$ in $N(\lambda)$. The method proposed here is a derivative from work by Ramani and Knight\cite{11} in developing a two-step component mode synthesis solution with convergence in large system eigensolutions.

The Method

To solve equation (1) for $\lambda$ and $\bar{q}$, the first order problem in equation (3) is first solved for all the eigenvalues and eigenvectors, and the eigenvectors are collected in a matrix $4$. Further, the eigenvectors are normalized such that
\[
\Phi^T \overline{N} \Phi = -I , \tag{4}
\]

where, \( I \) is the identity matrix. The transformation

\[
\overline{q} = \Phi q , \tag{5}
\]

is used for substitution in equation (1) and pre-multiplication by \( \Phi^T \) results in

\[
N(\lambda) q = 0 , \tag{6}
\]

where,

\[
N(\lambda) = \Phi^T \overline{N}(\lambda) \Phi \tag{7}
\]

Substituting for \( \overline{N}(\lambda) \) from equation (2) in equation (7) and using the resulting expression for \( N(h) \) in equation (6) results in

\[
[N_s + AN_s + \lambda^2 N_s + \lambda^2 N_s] q = 0 , \tag{8}
\]

where,

\[
N_s = \Phi^T \overline{N}_s \Phi \quad s = 0,1,2, \ldots r . \tag{9}
\]

Note that \( N_0 \) is the diagonal matrix of initial eigenvalues of equation (3). Also note that equation (6) is identical to equation (1) and there are no approximations involved, since all the eigenvectors of the basic linear eigenvalue problem were used to transform the coefficient matrices. Ramani and Knight\([11]\) used a perturbation approach to derive a convergence scheme operating on equation (6) such that the eigenvalues of the non-linear eigenvalue problem can be obtained by converging the eigenvalues of the basic linear eigenvalue problem. In doing so, the dependence between the eigenvalue and eigenvector was decoupled so that the resulting equations for convergence did not involve coupling between the eigenvalue and eigenvector.

Assume that approximate values \( \lambda_0 \) and \( q_0 \) that satisfy equation (6) are known. Then, in general, \( \lambda_0 \) and \( q_0 \) will not satisfy equation (6). Therefore, a correction \( \lambda_c \) to \( \lambda_0 \) and \( q_c \) to \( q_0 \) need to be determined that will satisfy equation (6). The perturbation approach expands the eigenvalue, the eigenvector and the matrix \( N(\lambda) \) in terms of a perturbation parameter \( \varepsilon \) as

\[
\lambda = \lambda_0 + \varepsilon \lambda_c ; \tag{10}
\]

\[
q = q_0 + \varepsilon q_c ;
\]

\[
N(A) = N_0 + \varepsilon N_1 + \varepsilon^2 N_2 + \varepsilon^2 N_3 + \ldots + \varepsilon^{r-1} N_r . \tag{11}
\]

Substituting the above expressions for \( \lambda, q \) and \( N \) in equation (6) and retaining only the terms with the first power of \( \varepsilon \) results in the basic equation from which the convergence scheme is derived, i.e.,

\[
N(\lambda_0) q_0 + \varepsilon N(\lambda_0) q_1 + \varepsilon \lambda_c N(\lambda_0) q_0 = 0 \tag{12}
\]

where, \( N'(\lambda) \) is given by the equation

\[
N'(\lambda) = N_s + \lambda N_s + \lambda^2 N_s + \lambda^2 N_s + \ldots + \lambda^{r-1} N_s . \tag{12a}
\]

Equation (11) represents a system of equations with one more unknown than the number of equations. This is the case with the solution of any eigenvalue problem because the eigenvector can be multiplied by an arbitrary scale factor. The problem was resolved by choosing to normalize the eigenvector, \( q \) as \( q^T q = 1 \).

The previous relations provide the basis from which iteration expressions are derived. Eigenvalue-eigenvector dependence is decoupled in the derivation.
of the iteration expressions. and this results in two iteration expressions for convergence. one each for the eigenvalue and eigenvector.

**Solution for the eigenvalue assuming that the eigenvector is known:** If, in equation (11), it is assumed that the eigenvector $q$ is known, i.e., $q_0 = q$ and $q_e = 0$, then

$$N(\lambda_0)q + \varepsilon \lambda_e N(\lambda_0)q = 0. \quad (13)$$

From this equation, the expression for the eigenvalue is derived in reference [11] by assuming terms in higher powers of $\varepsilon$ are small and then removing the $\varepsilon$ parameter to give

$$A = A_0 + q^T N(\lambda_0)q \quad (14)$$

In a iterative scheme,

$$\lambda^{k+1} = \lambda^k + q^T N(\lambda^k)q, \quad (15)$$

where the superscript on $\lambda$ denotes the iteration counter.

**Solution for the eigenvector assuming that the eigenvalue is known:** If it is assumed that the eigenvalue $\lambda$ is known, i.e., $\lambda = \lambda_0$ and $\lambda_e = 0$, then equation (11) becomes

$$N(\lambda)q_0 + \varepsilon N(\lambda)q_e = 0 \quad (16)$$

From this equation solve as in reference [11] for $q_e$ by retaining only terms with the first power of $\varepsilon$ and then removing $\varepsilon$ from the final expression gives

$$q_e = -(N_0 + \lambda N_0)^{-1} N(\lambda)q_0 \quad (17)$$

Therefore, the iteration expression for the eigenvector is

$$q^{(k+1)} = q^{(k)} - (N_0 + \lambda N_0)^{-1} N(\lambda)q^{(k)} \quad (18)$$

where the superscript on $q$ denotes the iteration counter. The diagonal structure of $(N_0 + \lambda N_0)$ enables easy computation of its inverse.

**Solution for the eigenvector and the eigenvalue starting from approximate initial values:** If both the eigenvalue and eigenvector are known approximately, the above iteration schemes can still be used, simultaneously converging both $A$ and $q$. Then, one step in an iteration involves the equations (15) and (18).

**Initial Values**

The initial values to start convergence for the $i^{th}$ eigenvalue and eigenvector are given by

$$\lambda_i^{(0)} = \Lambda(i) \quad (19)$$

$$q_i^{(0)} = \Lambda(i) \quad (20)$$

where $q_i^{(0)}$ has 1 in the $i^{th}$ position and $\Lambda$ is the diagonal matrix of initial eigenvalues. However, convergence cannot be started by evaluating the eigenvector $q_i^{(0)}$ since $[N_0 + \lambda_{ii}^0 N_i]$ is singular. Hence, to start convergence, the eigenvalue equation is iterated first and then the eigenvector. Thereafter, the set of equations (15) and (18) can be used consecutively for converging the eigenvalue and the eigenvector.

**Condensation**

The method described above may be computationally inefficient if the order of the matrices is large and only
the lowest few eigenvalues are desired. The inefficiency is caused by solving the basic linear eigenvalue problem for all its eigenvalues and eigenvectors. As in other eigensolving algorithms when only the lowest eigenvalues are needed the linear part of the eigenvalue problem only needs to be solved for its lowest eigenvalues. The number of such eigenvalues to be found depends on the final number of eigenvalues desired and cost considerations. In general, more initial eigenvalues provide more accurate final eigenparameters of the non-linear system. Then, the matrix Φ forms an incomplete set of eigenvectors of the linear system. The transformation given by equation (5) will now result in equation (6), but with matrices of order equal to the number of eigenvalues of the linear eigenvalue problem. The convergence scheme can now be applied to the resulting system of equations with much greater efficiency at some sacrifice of accuracy.

**Examples**

Two simple examples are included for demonstration. The first example solves the eigenvalue problem

\[(\overline{N}_0 + \lambda \overline{N}_1 + \lambda^2 \overline{N}_2)\overline{q} = 0,\]  

[21]

where,

\[
\overline{N}_0 = \begin{bmatrix}
3 & -1 & 0 & 0 \\
-1 & 3 & -2 & 0 \\
0 & -2 & 5 & -3 \\
0 & 0 & -3 & 4
\end{bmatrix},
\]  

[22]

\[
\overline{N}_1 = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -3 & 0 \\
0 & 0 & 0 & -4
\end{bmatrix},
\]  

[23]

\[
\overline{N}_2 = \begin{bmatrix}
-0.1 & 0.04 & 0 & 0 \\
0.04 & -0.12 & 0 & 0.02 \\
0.08 & -0.1 & 0.02 & 0
\end{bmatrix},
\]  

and

\[
\overline{N}_3 = \begin{bmatrix}
0 & 0 & 0.08 & -0.06
\end{bmatrix}
\]

A* eigenvalue problem like this could result from the use of dynamic stiffness matrices. With the matrix \(\overline{N}_0\) representing the static stiffness matrix, the matrix \(\overline{N}_1\) representing the negative of the mass matrix and the matrix \(\overline{N}_2\) representing the dynamic stiffness matrix associated with the second power of the eigenvalue. The results are reported in Table 1.

The eigenvalues of the basic linear system are reported in the first column as the initial eigenvalues. The second column shows the converged nonlinear eigenvalues along with the exact eigenvalues in parenthesis. The converged eigenvalues were obtained using the complete set of linear eigenvectors for transformation. The fourth eigenvalue could not be converged. Exact eigenvalues for this problem were obtained by forming block-companion matrices and solving the resulting linear eigenvalue problem. The third column shows the eigenvalues of the non-linear system using the condensed set of eigenvectors corresponding to the lowest three eigenvalues of the linear system for the transformation.

**Table 1. Results for Example 1**

<table>
<thead>
<tr>
<th>NO.</th>
<th>Initial Eigenvalues</th>
<th>Converged Eigenvalues/ (Exact)</th>
<th>Eigenvalues using Condensation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>0.19068</td>
<td>0.19042/(0.19042)</td>
<td>0.19042</td>
</tr>
<tr>
<td>2.</td>
<td>1.1760</td>
<td>1.1333/(1.1333)</td>
<td>1.1333</td>
</tr>
<tr>
<td>3.</td>
<td>1.9112</td>
<td>1.7958/(1.7958)</td>
<td>1.7983</td>
</tr>
<tr>
<td>4.</td>
<td>3.8887</td>
<td>* did not converge</td>
<td></td>
</tr>
</tbody>
</table>

For the second example, consider the problem

\[(\overline{N}_0 + \lambda \overline{N}_1 + \lambda^2 \overline{N}_2 + \ldots)\overline{q} = 0\]  

[25]

where
\[ \overrightarrow{N}_0 = \begin{bmatrix} 1.2405 & 0 & 0 & 0 & 0 \\ 0 & 0.4508 & 0 & 0 & 0 \\ 0 & 0 & 0.2530 & 0 & 0 \\ 0 & 0 & 0 & 0.0732 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (26)

\[ \overrightarrow{N}_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \] (27)

and

\[ \overrightarrow{N}_s(i,j) = -\sum_{r=1}^{3} \frac{m(i,r)n(j,r)}{\Lambda(r,r)^s} \quad s = 2, 3, 4. \] (28)

where,

\[ m = \begin{bmatrix} -0.1032 & 0.2426 & -0.0976 \\ 0.0180 & -0.0424 & 0.0171 \\ -0.0438 & 0.1028 & -0.0414 \\ 0.0158 & -0.0372 & 0.0150 \\ -0.0419 & 0.0985 & -0.0396 \end{bmatrix} \] (29)

\[ \Lambda = \begin{bmatrix} 0.9628 & 0 & 0 \\ 0 & 2.1737 & 0 \\ 0 & 0 & 2.6145 \end{bmatrix} \] (30)

This is an infinite-series eigenvalue problem that arises when two-step component mode synthesis (CMS) is performed as in reference [11]. The series will converge if the terms \( \lambda^s \overrightarrow{N}_s \) are decreasing in magnitude. In this case, the series converges as long as the eigenvalue is smaller than the smallest eigenvalue term in the \( \Lambda \) matrix.

The specific system in this case is one-dimensional, with 10 mass and 9 spring elements. For two-step CMS demonstration the system was broken into two components. The first two modes of each component were used in the first step of synthesis along with the rigid-body modes. For the convergence step, one additional mode of the first component and two additional modes of the second component were used.

The matrices given above are obtained during this process. The matrix \( \overrightarrow{N}_0 \) is already in the diagonal form and \( \overrightarrow{N}_1 \) is the negative of the identity matrix. Hence, the initial eigenvalues (of the linear problem) are the diagonal elements of \( \overrightarrow{N}_0 \) and are listed in the first column of Table 7.2.

The eigenvalues resulting from the convergence scheme and the exact eigenvalues are reported in the second column. Exact eigenvalues for a general infinite-series eigenvalue problem are difficult to determine, but because of the small size and full system model in this case, exact solutions could be obtained. For the fifth eigenvalue, the series becomes divergent as the lowest diagonal element of the matrix \( \Lambda \) is greater than the initial eigenvalue. Hence, the convergence scheme cannot find this eigenvalue.

### Table 2. Results for Example 2.

<table>
<thead>
<tr>
<th>NO.</th>
<th>Initial Eigenvalue</th>
<th>Converged Eigenvaues/ (Exact)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2.</td>
<td>0.07319</td>
<td>0.07319/(0.07319)</td>
</tr>
<tr>
<td>3.</td>
<td>0.25297</td>
<td>0.25240/(0.25240)</td>
</tr>
<tr>
<td>4.</td>
<td>0.45080</td>
<td>0.45043/(0.45035)</td>
</tr>
<tr>
<td>5.</td>
<td>1.24050</td>
<td>--* /0.93375</td>
</tr>
</tbody>
</table>

* could not find

Conclusions

A method for solving non-linear eigenvalue problems
arising in structural dynamics has been developed and demonstrated on two simple examples. The method is applicable in the solution of positive semidefinite non-linear eigenvalue problems. In general, the method proved successful in finding both the eigenvalues and eigenvectors with precision. The method can be employed in eigensolution using dynamic finite elements.

The method is an alternative to the companion-matrix method which is widely used to solve non-linear eigenvalue problems. A significant advantage of the method over the companion-matrix method is the preservation of bandwidth in large finite element models. Because of rearrangement of the coefficient matrices in the companion-matrix method, advantages of banded storage is lost and there is also an increase in the order of the system to be solved. However, the present method does not require rearrangement of the coefficient matrices, thus preserving bandwidth. Also, the order of the problem is not increased. The method also works successfully for infinite-series eigenvalue problems which cannot be solved using the companion-matrix method.

References


