ABSTRACT. Many activities in modal analysis research and applications are classified under the general area of the inverse problem. Examples are modal parameter identification, parameter identification, finite element model updating and damage detection, among others.

These type of problems usually result in a system of linear equations that are characterized by one or both of the following:

(a) Ill conditioned system of equations,
(b) Underdetermined system of equations.

Matrix decomposition or factorization techniques can be extremely helpful in dealing with or detecting singularities and numerical ill conditioning. Among these methods are:

(a) QR and QL algorithms,
(b) LU factorization,
(c) Cholesky decomposition,
(d) Singular Value Decomposition.

This paper is to introduce the theoretical grounds of these mathematical tools. Application of these techniques in modal analysis is illustrated and limitations and usefulness of solutions are emphasized.

1. INTRODUCTION AND PROBLEM DEFINITION

In many aspects of modal analysis, especially in the inverse problems, the formulation often reduces to a set of linear algebraic equations of the form:

\[ [A]x = b \]  \hspace{1cm} (1)

where \([A]\) is a known \(m \times n\) matrix which is called the data kernel, \(b\) is a known vector of length \(m\) and \(x\) is a vector of \(n\) unknowns.

For analysis problems, the matrices \([A]\) and \([b]\) are accurately and precisely known. In inverse problems, however, there exist many uncertainties and inaccuracies in both the data and formulation. Prior to addressing the existence of solution, it is important to define the rank of \([A]\). The rank, \(r\), of \([A]\) is the number of linearly independent rows or columns and

\[ r \leq \min(m, n) \]  \hspace{1cm} (2)

The general solution of equation \(1\) is:

\[ x = [A]^+ b \]  \hspace{1cm} (3)

where \([A]^+\) is the generalized inverse of \([A]\). For the inverse problem, in general, \(m > n\) and due to presence of noise \(r = n\). Here the matrix condition number is introduced.

1.1 Consistency of Equations and Existence of Solution

Consistency is here defined to describe equation \(1\) if this equation has at least one solution. In other words, due to erroneous formulation of equations or due to errors in \([A]\) and \([b]\), the vector \(b\) cannot be a linear combination of the columns of \([A]\). To systematically define consistency, let \(r'\) be the rank of the augmented matrix \([A : b]\) then:

(a) Equations are consistent if \(r' = r\)
(b) Equations are inconsistent if \(r' = r + 1\)
(c) \(m = n = r\) equations have a unique solution.
(d) If \(r = r'\) and \(m > n\) the equations have \((m - r)\) solutions.

1.2 Relation Between Rank and Number of Equations

(a) Even determined when \(m = n = r\)
(b) Under determined when \(r = r'\) and \(m > n\)
(c) Over determined when \(r < r'\)

The general solution of equation \(1\) is:

\[ x = [A]^+ b \]  \hspace{1cm} (3)

where \([A]^+\) is the generalized inverse of \([A]\). For the inverse problem, in general, \(m > n\) and due to presence of noise \(r = n\). Here the matrix condition number is introduced.

1.3 Condition Number of a Matrix

The condition number is a precise measure of a linear system's solution sensitivity to errors in both \([A]\) and \([b]\). If \(C(A)\) is defined as the condition number for a square matrix \([A]\),

\[ C(A) = ||A|| ||A^{-1}|| \]  \hspace{1cm} (4.1)

The error in the solution \(x\) can be as large as the errors in \([A]\) and \([b]\) magnified by \(C(A)\):

\[ \frac{||\Delta x||}{||x||} \leq C(A) \frac{||\Delta A||}{||A||} \]  \hspace{1cm} (4.2)

Thus it is clear that smaller condition numbers are sought after for more stable numerical solution.
1.4 The Least Squares Solution

This is one of the most used solutions in the inverse problem linear system solution. This solution is based on minimizing the length of the prediction error vector \( \{e\} \), where:

\[
\{e\} = \{A\} \{x\} - \{b\}
\]  

(5)

This gives an estimate of \( \{x\} \)

\[
\{x\} = [A]^T [A A^T]^{-1} \{b\} \quad r = m < n
\]  

(6.1)

\[
\{x\} = [A A^T]^{-1} [A]^T \{b\} \quad r = n < m
\]  

(6.2)

In this approach it is clear that if the condition number of \( [A] \) is large, the condition number of \( [A A^T] \) or \( [A^T A] \) is the square of the condition number of \( [A] \) and the resulting solution is more sensitive to noise.

For the above reason, matrix decomposition techniques offer an attractive alternative to the least squares solution.

2. MATRIX DECOMPOSITION TECHNIQUES

2.1 LU Decomposition

For a square matrix \( [A] \) it is possible to factorize \( [A] \) into a lower triangular matrix \( [L] \) and an upper triangular matrix \( [U] \) such that:

\[
[A] = [L][U]
\]  

(7)

Now to solve the equation

\[
[A]\{x\} = [L][U]\{x\} = \{b\}
\]  

(8)

first the equation

\[
[L]\{y\} = \{b\}
\]  

(9)

is solved using forward substitution, e.g. solving for \( y_1 \), then \( y_2 \ldots \) then \( y_n \), and then the equation

\[
[U]\{x\} = \{y\}
\]  

(10)

is solved using backward substitution.

2.1.1 LU Factorization of Rectangular Matrix

This is also possible and results in:

\[
[A] = \begin{bmatrix} [L] & [U] \end{bmatrix} \quad m > n
\]  

(10.1)

where \( [L] \) and \( [U] \) are \( n \times n \) lower and upper triangulars and \( [B] \) is \( (m - n) \times n \) full matrix; or

\[
[A] = [L][U \ B] \quad m < n
\]  

(10.2)

2.2 Cholesky Decomposition

For a square symmetrical matrix the LU factorization becomes:

\[
[A] = \begin{bmatrix} [L] & [L]^T \end{bmatrix}
\]  

(11)

3. THE QR AND QL DECOMPOSITION

A square matrix \( [A] \) can be decomposed into:

\[
[A] = [Q][R]
\]  

(12.1)

or

\[
[A] = [Q][L]
\]  

(12.2)

where \( [Q] \) is orthogonal \( ([Q]^{-1} = [Q]^T) \) and \( [R] \) is upper triangular while \( [L] \) is lower triangular \( (R \; \text{and} \; L \; \text{here are used for right or left of the matrix being nonzero}) \).

Now to solve

\[
[A]\{x\} = [Q][R]\{x\} = [Q][L]\{x\} = \{b\}
\]  

(13)

first the equation:

\[
[Q]\{y\} = \{b\}
\]  

(14)

is solved and

\[
\{y\} = [Q]^T \{b\}
\]  

(15)

then

\[
[R]\{x\} \; \text{or} \; [L]\{x\} = \{y\}
\]  

(16)

is solved using backward or forward substitution.

For rectangular matrices,

\[
[A] = [Q] \begin{bmatrix} [L] \\ \{o\} \end{bmatrix} \quad m > n
\]  

(17)

or

\[
[A] = [Q] \begin{bmatrix} [L] & [B] \end{bmatrix} \quad m < n
\]  

(18)

It is to be noted here that the condition number of \( [Q] \) is always 1 and the condition number of the cofactor is equal to condition number of \( [A] \).
4. SINGULAR VALUE DECOMPOSITION SVD

For any \( m \times n \) matrix \( [A] \) it can be shown that decomposition of \( [A] \) into

\[
[A] = [U][\Sigma][V]^T
\]  

(19)

can be performed, where \( [U] \) is \( m \times m \) and \( [V] \) is \( n \times n \) and \( [\Sigma] \) is \( m \times n \) containing the singular values (eigenvalues of \( [A] \)) along its "diagonal".

The following relations hold:

\[
[A][v]_i = \sigma_i^2 \{v\}_i, \quad i = 1, \ldots, r
\]

\[
[A]{v}\_i = 0 \quad i = r + 1, \ldots, n
\]

\[
\{v\}_i \{v\}_j = \delta_{ij} \quad i, j = 1, \ldots, n
\]

\[
[A][A]^T{u}\_i = \sigma_i^2 {u}\_i, \quad i = 1, \ldots, r
\]

\[
[A]^T{u}\_i = 0 \quad i = r + 1, \ldots, m
\]

\[
\{u\}_i \{u\}_j = \delta_{ij} \quad i, j = 1, \ldots, m
\]

\[
[A]{x}\_i = \sigma_i \{u\}_i, \quad i = 1, \ldots, r
\]

\[
[A]^T{u}\_i = \sigma_i \{v\}_i, \quad i = 1, \ldots, r
\]

Now considering the nonzero singular values as \( [\Sigma_r] \) the matrix \( [A] \) can be written as

\[
[A] = [U_r][U_r]^T = \sum_{i=1}^{r} \sigma_i \{u\}_i \{v\}_i^T
\]

As well the natural generalized inverse of \( [A] \) is

\[
[A]^+ = [V_r][\Sigma_r]^{-1}[U_r]^T = \sum_{i=1}^{r} \frac{1}{\sigma_i} \{v\}_i \{u\}_i^T
\]

4.1 Application of SVD for \( [A]{x} = \{b\} \)

Apparently SVD is a powerful and stable technique to solve the problem under consideration and for any situation: singular, ill conditioned, over determined or under determined.

This very characteristic of SVD renders it as a potential for misuse. Furthermore, and due to the nature of the problem or simple misformulation, the system of equations can be severely ill conditioned. In this situation many researchers tend to use regularization and truncate the "small" singular values the very of which could carry meaningful information.

The SVD solutions, when the matrix of coefficients is singular or when the system of equations is under determined, are purely mathematical solutions — the minimum norm solution — and could have little or no physical meaning and in some instances could violate the physics of the problem.

4.1.1 Numerical Illustration

This simple example may illustrate the case of underdetermined system of equations in an inverse problem. In an attempt to identify two masses and two stiffnesses the following equations were derived:

\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
m_1 \\
k_1 \\
m_2 \\
k_2
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
4
\end{bmatrix}
\]

The SVD solution for this case is \( 1 \ 1 \ 2 \ 2 \)^T. For another approach let the equations be

\[
\begin{bmatrix}
3 & -2 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
m_1 \\
k_1 \\
m_2 \\
k_2
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
4
\end{bmatrix}
\]

which gives an SVD solution of \( \{0.2938 \ -0.1538 \ 2. \ 2.\}^T \).

Needless to say, these two solutions are purely mathematical and relates nothing to the actual problem.

4.1.2 Use of SVD for Force Identification

Force identification is a modal analysis inverse problem which is inherently ill conditioned due to the inequality of the number of modes contributing to the measured response and the number, and locations, of the forces to be identified. The equations used here are based on identified FRF's matrix \( [H] \) and measured response vector \( \{x\} \).

\[
[H]\{F\} = \{x\}
\]

Here simulation and experimental cases are studied. For the simulation case, five modes are used to construct the FRF's at 8 locations on a simply supported beam. Responses for a known input force were calculated and noise was added to both the measured response and the FRF's. Table 1. shows the typical singular values of \( [H] \) at 43 and 120 Hz. A drop in the 6th singular value is noticeable; more distinctly at 120 Hz than the 43 Hz case. Figures (1.1) to (1.3) show some solutions imposed on the theoretical force. Figure 1.1 shows the solution using \( 8 \times 8 \) matrix without any regularization (using QR or SVD). The ill conditioned \( [H] \) magnifies the effects of measurements noise. Figure 1.2 shows the SVD solution truncating singular values above the 5th one. In this case, the minimum norm solution is different from the actual force. Figure 1.3 shows the solution for an \( 8 \times 5 \) \( [H] \) matrix (using full SVD or QR).
For a similar experimental case of force identification, the $[H]$ matrix is identified by curve fitting of four modes at 11 measurements. Figure 2 shows the change in singular values at five different frequencies. While $[H]$ is of rank 4, no clear drop in the fifth singular values were noticed. In fact it seems that the largest drop occurred between the 6th and 7th singular values. In this case as well, meaningful solutions can only be obtained for four forces.

<table>
<thead>
<tr>
<th>$i$</th>
<th>43 Hz</th>
<th>120 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.000e+00</td>
<td>1.000e+00</td>
</tr>
<tr>
<td>2</td>
<td>7.8043e-04</td>
<td>9.2507e-01</td>
</tr>
<tr>
<td>3</td>
<td>1.4649e-04</td>
<td>9.9090e-02</td>
</tr>
<tr>
<td>4</td>
<td>4.5725e-05</td>
<td>2.9108e-02</td>
</tr>
<tr>
<td>5</td>
<td>1.9126e-05</td>
<td>1.1693e-08</td>
</tr>
<tr>
<td>6</td>
<td>7.1644e-07</td>
<td>9.3552e-08</td>
</tr>
<tr>
<td>7</td>
<td>3.2414e-07</td>
<td>6.8775e-08</td>
</tr>
<tr>
<td>8</td>
<td>1.4955e-07</td>
<td>6.2403e-08</td>
</tr>
</tbody>
</table>

4.2 Use of SVD to Condense and Filter Data

Many activities in modal analysis involve the use of measured time domain or frequency domain data. The amount of data is usually large. SVD can be useful here to condense and filter data.

This is referred to as principal component analysis.

4.2.1 Numerical Illustration

Data is generated for ten measurements in both time and frequency domain, 301 time data points were used while 151 frequency measurements were considered. Noise was added to both sets of data at 20% rms ratio.
Each set of responses was factorized using SVD and since two modes simulated in the responses, a noticeable drop was observed at the third singular value.

The condensed response was constructed by truncating singular values above the second.

Storage wise the original time domain data required $301 \times 10$ storage locations. The filtered response required only $(301 \times 2 + 2 \times 10)$ for $[U]$ and $[S][V]^T$. For the frequency data storage was reduced from $1510$ to $(151 \times 2 + 2 \times 10)$.

Here, two questions need to be asked:

(a) Did SVD remove the noise from the data?

(b) Did the reconstructed data suffer any distortions due to SVD?

For the time domain data Figures 3.1 to 3.4 show the noisy time responses, the singular values, the filtered time response and the noise before and after SVD filtering. Figures 4.1 to 4.4 show the corresponding data for the frequency domain data. In both cases, the filtered data remain to be noisy even though the noise level is lower.

![Figure 3.1](image1.png)

**Figure 3.1** 10 time histories with 20% rms noise

![Figure 3.2](image2.png)

**Figure 3.2** Singular value

![Figure 3.3](image3.png)

**Figure 3.3** 10 time histories from truncated SVD

![Figure 3.4](image4.png)

**Figure 3.4** SVD noise

**Figure 3.** Results for Time Domain Data Principal Component Analysis
Table 2 shows the two singular values for the noisy and filtered data.

Table 2. Singular Values for Responses

<table>
<thead>
<tr>
<th></th>
<th>Time Data</th>
<th>Frequency Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>original</td>
<td>filtered</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>14.9346</td>
<td>15.0526</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>9.8180</td>
<td>9.8667</td>
</tr>
</tbody>
</table>

5. CONCLUSIONS

Matrix decomposition techniques are useful tools in solving systems of algebraic equations. They can produce better solutions for ill conditioned systems or, in the least, point out the systems' numerical deficiencies.

The use of these decomposition techniques, such as SVD with regularization, to solve seriously ill conditioned systems or underdetermined systems can, and usually, results in erroneous or nonphysical solutions and should be avoided.

REFERENCES