MAXIMUM LIKELIHOOD IDENTIFICATION OF MODAL PARAMETERS FROM OPERATIONAL DATA

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ABSTRACT

In the present contribution, the applicability of frequency-domain Maximum Likelihood (ML) identification techniques in the field of operational modal analysis is investigated. Attention is paid to the derivation of the spectral densities and noise information required for the ML estimator. The frequency-domain ML approach is applied to a real case study and confidence intervals are derived on the estimated parameters. Using these confidence intervals, it is possible to automate the model order selection.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>Sampling period,</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of time samples,</td>
</tr>
<tr>
<td>$P$</td>
<td>Number of segments,</td>
</tr>
<tr>
<td>$D$</td>
<td>Segment size,</td>
</tr>
<tr>
<td>$\hat{R}_{x_i} [m]$</td>
<td>Correlation functions,</td>
</tr>
<tr>
<td>$\hat{S}_{x_i} (j\omega)$</td>
<td>Spectral density.</td>
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</table>

1. INTRODUCTION

Present modal model identification methods and procedures are limited to forced excitation laboratory tests. However, during in-operation tests (either during actual operation or during operation-simulation tests), the real loading conditions on the product are present. They often differ significantly from the ones in laboratory testing. As all real-world systems are to a certain extent non-linear, the models obtained under real loading will be linearized for much more representative working points. Additionally, they will properly take into account the environmental influences on the system behavior (pre-stress of suspensions, load-induced stiffening, aero-elastic interaction, ...). Another interest in in-operation modal analysis stems from the fact that in many cases (small excitation of off-shore platforms, traffic/wind excitation of civil constructions) forced excitation tests are very difficult, if not impossible, to conduct, at least with standard testing equipment. Operating data are often the only ones available. Furthermore, the availability of in-operation established models opens the way for in situ model-based diagnosis and damage detection. Hence, a considerable interest exists in extracting valid models directly from operating data. Finally, one may observe that in many cases (e.g., car road tests and aircraft/spacecraft flight tests), large in-operation data sets are measured anyway, for level verification, for operating field shape analysis and other purposes. Hence, extending classical operating data analysis procedures with modal parameter identification capabilities will allow a better exploitation of these data.

A well-adopted way of dealing with operational analysis in industry is based on a peak-picking technique applied to the power- and cross-spectral densities of the operational responses [1]. By selecting the peaks in the spectra, approximate estimates for the resonance frequencies and operational deflection shapes can be obtained. These shapes can then be compared to or even decomposed into the laboratory modal results. Correlation of the operating data set with the modal database measured in the lab allows to assess the modes, dominant for a particular operating condition. In case of partially correlated inputs (e.g. road analysis), principal component techniques are employed to decompose the multi-reference problem into subsets of single reference problems, which can each be analyzed in parallel [2]. These decomposed set of data can be fed to an animation program, to interpret the operational deflection shapes for each principal component as a function of frequency [3]. The power- and cross-spectral density peak-picking method requires a lot of engineering skill to select the peaks which correspond to system resonances. Additionally, it does not provide any
information about the damping of the modes and the operational deflections shapes may differ significantly from the real mode shapes in case of closely spaced modes. Pre-knowledge of a modal model derived from FRF measurements in the lab is often indispensable to successfully perform a conventional operational analysis. Therefore, curve-fitting techniques which would allow to extract the modal parameters from the operational data would be of a great use for the engineer. Such techniques would allow to identify the modes which are dominantly excited in driving condition and this information might even be used to optimally perform some traditional FRF tests in the laboratory.

It has been demonstrated that correlation-driven stochastic subspace techniques and (polyreference) LSCE applied to correlation functions instead of impulse response functions yield promising results for output-only system identification [4]. Both methods are time domain methods and have some benefits and drawbacks. Both require the selection of the model order. In practice, stabilization diagrams showing the stability of the poles as function of increasing model order are used to distinguish the spurious modes from the physical ones. The correlation-driven stochastic subspace approach directly yields the natural frequencies, damping ratios and mode shapes. Due to its mechanism of truncation of singular values, less overspecification of the model order is needed and the stabilization diagrams are easier to interpret than the LSCE method, especially when the number of references is, say more than 4 in the LSCE method. The computational load is however significantly higher than LSCE and practically speaking, the method has difficulties to handle more than 100 DOFs on an average modern workstation. The LSCE technique overcomes this limitation. A fast implementation is possible, which is capable of dealing with large amounts of DOFs (up to 1000 and more DOFs is no exception). However, the LSCE method does only yield global estimates of the natural frequency and the damping ratios. The mode shapes need to be derived in a second step by curve-fitting the auto- and cross-correlations or the power­spectral densities in a least-squares sense. Due to the noisy nature of the data as the measurements were performed during operation and no inputs could be measured, this step is often far from straightforward and requires some skill of the engineer. By overlaying the actual test data with the synthesized data, the least-squares curve-fitting step provides a graphical quality check of the modal model. Both methods do not deliver confidence intervals on the estimated modal parameters, which would be a great help for in-operation monitoring and diagnostics of vibrating structures [5].

In the present contribution, the applicability of frequency-domain Maximum Likelihood (ML) identification techniques in the field of output-only modal analysis is investigated. It has been recognized that maximum likelihood parameter estimation techniques have the potential to be a significant breakthrough in flight flutter testing [6]. This is because flight flutter data are quite noisy while the ML estimator, which is based on a statistical approach [7], can take noise information into account (e.g., the variance of the noise on the measured FRFs). The same conclusion remains valid for operational modal analysis in general. Because operational tests are performed during operation, the measurements can be rather noisy. Moreover, no force measurements are available which complicates the extraction of the modal parameters. Contrary to the LSCE method and the stochastic subspace technique, an important advantage of the ML approach is the possibility to generate confidence intervals for all estimated parameters. Also, the improved accuracy because the noise information is taken into account is an important asset. Moreover, the ML algorithm has been optimized to handle large amounts of data within a reasonable computation time [8].

2. THEORY ASPECTS

2.1 Modal decomposition of spectral densities

Consider an $n$-DOF representation of a mechanical system. Let ($f(t)$) denote the (unknown) force input vector of size $n$ at continuous time $t$ and ($x(t)$) the output displacement vector of size $n$. Let the $nxn$ FRF matrix between the outputs and inputs be given by

$$ [H(j\omega)] = \begin{bmatrix} H_{11}(j\omega) & \cdots & H_{1n}(j\omega) \\ \vdots & \ddots & \vdots \\ H_{n1}(j\omega) & \cdots & H_{nn}(j\omega) \end{bmatrix} $$

(1)

According to the modal theory of mechanical systems, the FRF matrix can be decomposed as follows

$$ [H(j\omega)] = \sum_{r=1}^{N_m} \left( \frac{[\psi_r]_r^T [P_r]}{j\omega - \lambda_r} + \frac{[\psi_r^T [Q_r]_r}{j\omega - \lambda_r^*} \right) $$

(2)

where $N_m$ is the number of modes, $\lambda_r$ denotes the $r$-th pole, $[\psi_r]$ is the $r$-th mode shape and $[P_r]$ is the participation vector of the $r$-th mode.

For stationary stochastic processes ($f(t)$), the $nxn$ spectral density matrix of the outputs, $[S_{xx}(j\omega)]$, is given by

$$ [S_{xx}(j\omega)] = [H(j\omega)] [S_y(j\omega)] [H^T(j\omega)] $$

(3)

where $[S_y(j\omega)]$ is the spectral density matrix of the input forces. Substituting (2) in (3) and assuming white noise inputs, it is easily shown that the spectral density matrix of the outputs can be modally decomposed as follows

$$ [S_{xx}(j\omega)] = \sum_{r=1}^{N_m} \left( \frac{[\psi_r]_r^T [Q_r]}{j\omega - \lambda_r} + \frac{[\psi_r^T [Q_r]}{j\omega - \lambda_r^*} \right)$$

(4)
where \( \{Q\} \) is the reference vector of the \( r \)-th mode. This reference vector is a complex function of the spectral density matrix of the random input force and the modal parameters of the mechanical system.

Equation (4) forms the basis for frequency-domain output-only modal analysis. Note that the order of the model in (4) is twice the order of (2).

Taking the discrete-time inverse Fourier transform of equation (4) yields the correlation matrix \( \mathbf{R}[m] \)

\[
\mathbf{R}[m] = \sum_{r=1}^{N} \{\psi_r\}_{[m]} e^{j\lambda_r m T} + \{\psi_r\}_{[m]}^{*} e^{-j\lambda_r m T} \quad \text{for } m \geq 0
\]

\[
\sum_{r=1}^{N} \{\psi_r\}_{[m]} e^{j\lambda_r m T} + \{\psi_r\}_{[m]}^{*} e^{-j\lambda_r m T} \quad \text{for } m < 0
\]

where \( T \) is the sampling period. Interesting to note is that the causal part of the correlation functions (positive lags) gives rise to the two first terms in (4) while the two last terms in equation (4) can be explained by the non-causal part of the correlation functions. Equation (5) corresponds to the basic idea of the NExT principle which says that the correlation functions can be expressed as a sum of decaying sinusoids [9]. Each decaying sinusoid has a damped natural frequency and damping ratio that is identical to the one of the corresponding structural mode. The LSCE method and the correlation-driven stochastic subspace technique only use the causal part of the correlation functions to extract the modal parameters. In [13] the correlogram approach turned out to yield more accurate estimates than the classical periodogram method.

### 2.2 Estimation of spectral densities

When one wants to use equation (4) to identify the modal parameters of the system from output-only data, it is important that accurate estimate of the spectral density functions are obtained from finite sequences of measured time samples. Basically, two classical approaches exist to estimate the spectral densities [10].

The direct method, or periodogram, operates directly on the data set to yield a spectral density estimate. On the other hand, the indirect method (i.e., the correlogram approach) first estimates correlation functions, and then discrete-time Fourier transforms it to obtain the spectral density estimates. For both approaches, the user is faced with several trade-offs in an effort to produce statistically reliable spectral estimates of highest possible resolution from a finite amount of data samples.

Two quantities can be used for measuring the quality of the estimate: the estimation bias and variance. A usual trade-off between the variance and bias error needs to be made. Both will have an effect on the accuracy of the modal parameters.

#### 2.2.1 The correlogram approach

The unbiased discrete-time correlation estimate between two output signals \( x[k] \) and \( x[k] \) of length \( N \) is given by

\[
\hat{R}_{xx}[m] = \left\{ \begin{array}{ll}
\frac{1}{N-m} \sum_{k=0}^{N-1} x[k+m] x[k] & \text{for } 0 \leq m \leq N-1 \\
\frac{1}{N-m} \sum_{k=0}^{N-1} x[k+m] x[k] & \text{for } -(N-1) \leq m < 0
\end{array} \right.
\]

The biased correlation estimate uses \( 1/N \), rather than \( 1/(N-m) \). The FFT algorithm can be used to obtain a fast implementation of the convolution [11].

The spectral density matrix is obtain by Fourier transforming (6)

\[
\hat{S}_{xx}(j\omega) = T \sum_{m=-L}^{L} w[m] \hat{R}_{xx}[m] \exp(-j\omega m T) \quad (7)
\]

where \( w[m] \) is an odd-length \((2L+1)\)-point lag time window (e.g., Hamming) over the interval \(-L \leq m \leq L \). The maximum lag index \( L \) is typically much less than the number of data samples in order to avoid the greater statistical variance associated with the higher lags of the correlation estimates [10].

Two important remarks need to be made when the correlogram estimator is used for the purpose of modal parameter estimation:

- Only the positive lags in equation (7) are required as they contain all the information needed to identify the modal parameters. This results in

\[
\hat{S}_{xx}^+(j\omega) = T \sum_{m=0}^{L} \hat{S}_{xx}^+(j\omega) \exp(-j\omega m T) \quad (8)
\]

- To reduce noise as well as leakage effects, the use of an exponential time data window

\[
w[m] = e^{-\gamma |m|}
\]

is preferred as it offers the advantage that the extracted poles can be corrected for the \( \beta \)-factor of the time window

\[
\lambda_r \text{corrected} = \lambda_r \text{estimated} + \beta
\]

#### 2.2.2 Derivation of the noise information

The basic idea to derive the required noise information (i.e., the variances of the correlograms) is to divide the data sequence of \( N \) samples into \( P \) non-overlapping segments of \( D \) samples each, such that \( DP \leq N \). For each segment \( s \) (s = 0, ..., \( P-1 \)), the correlograms \( \hat{S}_{xx}^+(j\omega) \) are computed. If the \( P \) segment correlograms are statistically independent, which is approximately true if the correlation function is
small for lags \( m > D \) and no or small overlap between the
segments, then
\[
\hat{S}_{x_i x_j}^*(j\omega) = \frac{1}{P} \sum_{p=0}^{P-1} \hat{S}_{x_i x_j}^*(j\omega)
\]
(11)

may be interpreted as the sample mean of a set of \( P \)

independent observations. An estimate for the variance is
then given by
\[
\text{var}(\hat{S}_{x_i x_j}^*(j\omega)) = \frac{1}{P} \left( \frac{1}{P} \sum_{p=0}^{P-1} \left| \hat{S}_{x_i x_j}^*(j\omega) \right|^2 - \left| \hat{S}_{x_i x_j}^*(j\omega) \right|^2 \right)
\]
(12)

The variance will decrease with increasing \( P \). The bias error
due to leakage increases for decreasing \( D \). For fixed \( N = PD \), there is a usual trade-off between high spectral
resolution (\( O \) is as large as possible) and minimizing the
estimation variance (\( P \) is as large as possible).

3. THE ML APPROACH

In [8], a frequency-domain Maximum Likelihood (ML)
estimator is proposed to estimate the model parameters
from frequency response measurements. In the present
paper, it will be shown that a similar approach can be used
to derive the modal parameters from spectral density functions. Before doing so, the basic equations of the ML
solver for frequency response measurements are briefly recapitulated [8].

3.1 Frequency response measurements

The Frequency Response Functions (FRFs) are modeled as
\[
\hat{H}_{oi}(\omega_f) = \frac{N_{oi}(\omega_f)}{D(\omega_f)}
\]
(13)

for \( i = 1, \ldots, N_i \) and \( o = 1, \ldots, N_o \) with
\[
N_{oi}(\omega_f) = \sum_{j=0}^{n} \Omega_j(\omega_f) B_{nj}
\]
(14)

the numerator polynomial between output \( o \) and input \( i \) and
\[
D(\omega_f) = \sum_{j=0}^{n} \Omega_j(\omega_f) A_j
\]
(15)

the common-denominator polynomial. The polynomial basis
functions \( \Omega_j(\omega_f) \) are given by \( \Omega_j(\omega_f) = \exp(-i\omega_f T \cdot j) \)
(i.e., a discrete-time model with \( T \) the sampling period is
used). The coefficients \( A_j \) and \( B_{nj} \) are the parameters to
be estimated (they will be represented by the parameter vector \( \theta \) in the sequel of this section).

Assuming the measured FRFs, \( H_{oi}(\omega_f) \), to be (complex) normally distributed and mutually uncorrelated, the
(negative) log-likelihood function reduces to [7, 8]
\[
\ell_{\text{ML}}(\theta) = \sum_{a=1}^{N_a} \sum_{o=1}^{N_o} \sum_{f=1}^{N_f} \frac{\left| \hat{H}_{ao}(\theta, \omega_f) - H_{ao}(\omega_f) \right|^2}{\text{var}(H_{ao}(\omega_f))}
\]
(16)

The ML estimate of \( \theta \) (i.e., the polynomial coefficients) is
obtained by minimizing (16). This can be done by means of
a Gauss-Newton optimization algorithm, which takes
advantage of the quadratic form of the cost function (16).
Instead of minimizing (16), the following “MLE-like”
(complex) logarithmic estimator [8, 12] can be used
\[
\ell_{\text{LOG}}(\theta) = \sum_{a=1}^{N_a} \sum_{o=1}^{N_o} \sum_{f=1}^{N_f} \frac{\left| \text{log}(\hat{H}_{ao}(\omega_f)/H_{ao}(\omega_f)) \right|^2}{\text{var}(H_{ao}(\omega_f))/|H_{ao}(\omega_f)|^2}
\]
(17)

This logarithmic estimator has nearly MLE properties and
has been proven to be more robust than the ML solver to
tack of prior noise information and outliers [12].

If no noise information is available, the unweighted version
has to be used [7]
\[
\ell_{\text{LOG}}(\theta) = \sum_{a=1}^{N_a} \sum_{o=1}^{N_o} \sum_{f=1}^{N_f} \left| \text{log}(\hat{H}_{ao}(\omega_f)/H_{ao}(\omega_f)) \right|^2
\]
(18)

This estimator is less (statistically) efficient [7] than (17).

3.2 Output-only measurements

The approach we follow here consists in replacing the
FRFs by a quantity that can be derived from output-only
measurements such as for instance spectral densities
spectra (in analogy with the LSCE approach). As explained
in Sec. 2.1 the spectral densities can be modeled by means of
transfer functions (4). Consequently, it is possible to
apply the frequency-domain ML estimator to derive the
modal parameters (with the exception of the modal
participation factors). In Sec. 2.2, several approaches are
outlined to derive the spectral densities as well as their
variance. Sufficient attention should be paid during the
derivation of the spectral densities to the reduction of errors
such as leakage. Interesting to note is also that the use of
(8) instead of the classical spectral density (7) reduces the
order of the denominator polynomial with a factor of 2.
Consequently, the computational load is also seriously reduced. Also, the application of an exponential window in
the correlogram method allows to correct the estimated
modal parameters for the bias error. These two
considerations make that the correlogram approach is
preferred to the periodogram method for the extraction of
modal parameters from output-only data [13].
4. APPLICATION TO A STEEL SUBFRAME STRUCTURE

A validation test was performed on a breadboard model of a steel frame in order to assess the usefulness of the ML estimator for modal parameter extraction from output-only data. The structure resembles the subframe of a car to be connected to the body at 4 locations and on which the engine has to be mounted. Figure 1 shows the FE geometry of the subframe. The frame is approximately 720 mm long and 170 mm wide and the weight is about 9.8 kg. The frame was suspended on four flexible threads and the responses were measured in the vertical direction at 27 points for vertical excitation at 2 points. The shaker attachment points are indicated in Figure 1.

![FE geometry of the subframe test structure with indication of the 2 excitation locations.](image)

Using dual random shaker excitation, time records of the 2 inputs and the 27 outputs were measured, sampled at 1024 Hz. Several analyses were performed in order to assess the effects of the variance and the bias errors of the estimated spectral densities on the modal parameters. For each analysis, 32K samples were used to estimate the spectral densities. The analyses were done for the frequency range 0–512Hz and the number of modes which needed to be specified for the ML estimator equaled 20. The power- and cross-spectral densities between all 27 outputs and only 2 outputs serving as references were fed to the ML algorithm. A frequency-domain least-squares estimator [8] was used to get the initial values for the ML estimator and the number of iterations in the Gauss-Newton optimization algorithm equaled 20. The logarithmic version (17) of the estimator was used. Two main groups of analyses, with and without exponential window, were performed:

<table>
<thead>
<tr>
<th>Analysis</th>
<th>segment size D</th>
<th>time window W</th>
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</thead>
<tbody>
<tr>
<td>E1</td>
<td>512</td>
<td>Exp.</td>
</tr>
<tr>
<td>E2</td>
<td>2048</td>
<td>Exp.</td>
</tr>
<tr>
<td>R1</td>
<td>512</td>
<td>Rect.</td>
</tr>
<tr>
<td>R2</td>
<td>2048</td>
<td>Rect.</td>
</tr>
</tbody>
</table>

Measured and synthesized spectral densities are given in Figure 2. One can notice that the analyses with exponential window are of better quality. This is because the exponential window attenuates the effect of the larger statistical variance associated with the higher lags of the unbiased correlation function. The results for the analyses with exponential window are summarized in Table 1.

To separate the physical poles from the mathematical ones, a stabilization diagram can be used as illustrated in Figure 3 for the E2 data and using a frequency-domain least squares estimator [8]. Almost no mathematical poles are present in Figure 3 which simplifies the model order selection. These results can be used as starting values for the ML estimator. It is however also possible to use the ML estimator in a 'batch' mode. In that case, no stabilization diagram is required to select the physical modes: all modes are passed to the ML estimator. The selection is done at the end of the ML iterations and can be automated by taking the uncertainties into account of the estimates. In Figure 4 the standard deviations on the resonance frequency of all estimated modes are given. One notes that the uncertainty of the physical modes is 100 times smaller than the uncertainty of the spurious modes. In both cases (i.e., by using the stabilization chart or by using the uncertainties on the estimates) the same modes were selected.

5. CONCLUSIONS

In this paper, it has been shown how the frequency-domain ML estimator can be used to extract the modal parameters from output-only data. Instead of using FRFs, the ML estimator is fed by spectral density functions. The correlogram method is preferred for accurate estimation of the damping ratios. To reduce the required modal order, it is recommended to discrete-time Fourier transform only the positive time lags of the correlation functions. Exponential windowing can be used to decrease the effect of the noise (at the higher time lags) as well as the leakage errors. The modal parameters can be corrected for the effect of the exponential window. The results obtained with the ML estimator are very promising. Confidence levels on the modal parameters are provided, which might be of great interest for damage detection applications. It also allows to automate the model order selection.

REFERENCES

Figure 2: Measured spectral densities (+) and synthesized model (solid line). Analysis E1 (a), E2 (b), R1 (c), and R2 (d).


Figure 3: Stabilization chart obtained with the least-squares frequency-domain pole estimator [8].

Figure 4: Standard deviation of the estimated resonance frequencies.

Table 1: Estimation results together with the standard deviations.

<table>
<thead>
<tr>
<th>Mode no.</th>
<th>Damped resonance frequencies</th>
<th>Damping ratios</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>Analysis E1</td>
<td>Analysis E2</td>
</tr>
<tr>
<td></td>
<td>f (Hz)</td>
<td>std (Hz)</td>
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<tr>
<td>1</td>
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<td>2</td>
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