FINITE ELEMENT FORMULATION FOR VIBRATION ANALYSIS OF MULTI-LAYER LAMINATED BEAM

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ABSTRACT
A new approach is developed for finite element modeling of multi-layer laminated beams. Within the beam element, the laminate is decomposed as multiple basic layers, such as base beam layers of metal like, soft-core layers of adhesives, and soft coating layers of polymers, and mass and stiffness matrices of these basic layers are assembled together to generate those matrices for the beam element. By associating displacements of the soft layers to those of base beam layers, the method overcomes the aspect ratio limitation which is always a problem for the conventional finite element method because soft layer is usually much thinner than base beam layers in the laminates. In the conventional finite element method, modal strain energy method is usually used for structural damping analysis, which becomes quite inaccurate when the damping of the structure becomes high. This method allows the direct consideration of complex modulus of any layer so that it offers much more accurate damping analysis for such structures.

NOMENCLATURE

\[ \begin{align*}
    E & \quad \text{Young's modulus} \\
    G & \quad \text{shear modulus} \\
    \nu & \quad \text{poison ratio} \\
    T & \quad \text{kinetic energy} \\
    V & \quad \text{potential energy} \\
    \delta & \quad \text{variation operator} \\
    [M_e] & \quad \text{element mass matrix} \\
    [K_e] & \quad \text{element stiffness matrix}
\end{align*} \]

1. INTRODUCTION
In the practice of noise and vibration control, the application of damping treatments to the structure is playing more and more important role. A lot of different methods have been developed to design structures with maximum possible damping capacity. There is a variety of damping mechanisms that can be exploited in applied damping treatments. Significant damping of flexural vibration can be obtained from interfacial slip at contact surface, from fluid viscosity, from friction in fibrous materials, and from the distortion of dissipative, rubber-like materials. The most effective approach to date is perhaps the application of viscoelastic damping materials, and numerous successful case applying viscoelastic damping to control structural noise and vibration can be found in applications from aerospace structures to hard disk drives. In the past 50 years, considerable effort has been devoted to the studies of dynamic characteristics of viscoelastically damped structures [1].

The earliest work on damping analysis can be found mostly related to the viscoelastic material property characterization, because the knowledge of the complex modulus is essential for the efficient use of these materials. To develop an understanding of the parameters in the constrained layer damper, Ross, Kerwin and Unger [2] outlined the dominant design parameters for the case where all layers vibrate with the same sinusoidal spatial dependence. The outer layers are assumed to deform as
Eular-Bernoulli beams and the adhesive is assumed to deform only in shear, which leads to a single fourth order beam equation where the equivalent complex bending stiffness depends on the properties of the three layers. The beauty of the fourth order equation is not only in that one can estimate the natural frequencies and loss factors of the laminate providing that one knows the complex properties of the viscoelastic material, but also in that it can be used in the inverse problem. By knowing the natural frequencies and loss factors of the laminate, one can estimate the complex material properties analytically using fourth order equation, which has been used for decades in identification of viscoelastic material complex modulus. The problem with the fourth order equation is that it is derived from simply supported-simply supported boundary conditions which is not always true in real world application, and it neglects the thickness deformation of the viscoelastic layer which has been shown that it is not very accurate for flexible cores. To extend Ross, Ungar and Kerwin's analysis to beams with general boundary conditions in which sinusoidal spatial dependence cannot be assumed, Mead etc. obtained a sixth order equation of motion. It is assumed that the beam's deflection is small and uniform across a section, the axial displacements are continuous, the base and constraining layers bend according to the Eular hypothesis, the damping layer deforms only in shear, and the longitudinal and rotary inertia effects are insignificant. The validity of the analysis is therefore limited to some upper range of core stiffness. Miles etc. obtained a sixth order equation of motion by using Hamilton's principal. The assumptions were equivalent to those of Mead except that relative transverse deflection is permitted between the outer layers and longitudinal inertia is included. The analysis neglects rotary inertia in all layers and shear deformation in the outer layers, which limits the applicability to the lower order, long wavelength modes. However, the errors introduced by these simplifications may be neglected when the wavelength is more than about four times the thickness of the thickest outer layer.

Though analytical methods are useful for predicting damping characteristics of some simple structures such as beams and plates, finite element approach remains to be the method of choice when complex physical systems have to be analyzed. Under the framework of finite element analysis, several different modeling techniques have been developed to deal with damped structures. A common approach to model multiple layer structure is to use shell element with equivalent cross sectional properties. This approach has a drawback on boundary condition handling because it treats all layers as one layer so any boundary condition applies to all layers. This is normally not true for constrained layer damping treatment where the constrained layer usually has free boundary condition while the base structure might have any kind of boundary condition. To estimate damping loss factors associated with each mode, modal strain energy method is widely used based on the real eigven vectors obtained from finite element analysis of the corresponding undamped structure. In the modal strain energy method, the strain energy in each layer is calculated using the modal shape, and then the dissipative energy is calculated proportional to the strain energy in the damping layer and the material loss factor. The modal loss factor is finally obtained by calculating the ratio of the dissipative energy to the total structural energy. There are some limitations, however, in the analysis of the damped structures using conventional finite element method. For achieving the best damping result, the thickness of the damping layer is normally very thin, which results in the dramatic increase of element density used in the finite element modeling to get a proper aspect ratio for accurate prediction. Modal strain energy method becomes quite inaccurate when the damping of the structure becomes high because it is always assumed that the structural modal shapes are real while actual modal shapes of the damped structures are quite complex. Xu etc. derived a new finite element type, compound beam element, for sandwich beam by considering the complex modulus of the adhesive layer and the effects of both shear and thickness deformation in the adhesive layer.

In this paper, the above mentioned approach is further applied to finite element analysis of multi-layer laminated beam. The laminated beam is modeled using a 1-D beam element. Within the beam element, the laminated beam is decomposed as multiple basic layers, such as base beam layers of metal like, soft-core layers of adhesives, and soft coating layers of polymers. The equation of motion for the vibration of a laminated beam is obtained using Hamilton's Principle and the mass and stiffness matrices of these basic layers are derived. The laminated beam element mass and stiffness matrices are obtained by assembling together those matrices for each layer. Some assumptions are made in the derivation such that the deformation of soft layers can be determined by the displacement of the base beam layers. The use of the compound beam element results in significant simplification of sandwich beam modeling to a 1-D model instead of a 2-D model as in the conventional method, which dramatically reduces the amount of elements used while maintaining the desired accuracy. This method allows the direct consideration of complex modulus of any layer so that it offers much more accurate damping analysis for such structures.

2. DERIVATION OF MULTILAYER LAMINATE ELEMENT

For the derivation of the multi-layer laminated beam element, a segment of the beam is illluminated in Figure 1. Multi-layer laminated beam is usually composed of layers of stiff basic beam, normally metal, and layers of soft materials such as adhesive and polymer, etc.

In Figure 2, a multi-layer sandwich beam is decomposed to some basic components. These components include a single layer base beam as shown in (a), a constrained damping layer added to the base beam as in (b), and a free damping layer added to the base beam as in (c).
The equation of motion for vibration of a laminated beam will be obtained using Hamilton's Principle. The deformation to be considered will only include the transverse and longitudinal displacements as well as the rotation of both upper and lower layer base beams. It will be shown that the deformation of the middle, adhesive layer will be determined by displacements of both base beam layers. Hamilton’s Principle states:

\[ \delta \int_T T - V + W_0 dt = 0 \]

where \( T \) is the kinetic energy, \( V \) is the potential energy, \( W \) is the work done by external forces, and \( \delta \) is the variation operator. When the system under study is a continuum, \( T \) and \( V \) are given by integration over the volume of the body. In the following it is assumed that no external forces are acting on the body so that \( W \) is set to zero. Since we are primarily interested in the energy dissipation, or damping, by the viscoelastic adhesive layer, its dissipative abilities will be included in the model by considering the shear modulus, \( G \), and Young's modulus, \( E \), of the adhesive, to be complex. The kinetic energy and potential energy for the three basic components in Figure 2 will be derived in the following discussion.

For the single base layer as shown in (a), the kinetic energy may be written as,

\[
T_s = \frac{1}{2} \int \left[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) + \frac{1}{2} \mathbf{u} : \mathbf{D} \mathbf{u} \right] \, dt
\]

and the potential energy can be expressed as,

\[
V_s = \frac{1}{2} \int \left[ \frac{E_s}{2} \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) + E_s \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \right] \, dt
\]

For the constrained damping layer as shown in Figure 2 (b), kinetic energy and potential energy in the base beam layer and the constraining layer can be expressed in the same way as for the single base beam layer by replacing \( w_b, u_b, \phi_b \) and \( \delta_b \) in equations (2), (3) with \( w, u, \phi \) for the constraining layer. Kinetic energy and potential energy in the damping layer will be derived from the displacements of both base beam layer and constraining layer as follows. The kinetic energy can be expressed as,

\[
T_d = \frac{1}{2} \int \left[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) + \frac{1}{2} \mathbf{u} : \mathbf{D} \mathbf{u} \right] \, dt
\]

\[
V_d = \frac{1}{2} \int \left[ \frac{E_d}{2} \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) + E_d \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \right] \, dt
\]

The contribution to the potential energy due to the extension of the adhesive in the thickness direction, which can be considered as plane strain, and the extension in longitudinal direction, which is plane stress, is given by the integration over the difference between base beam layer and the constraining layer,

\[
V_d = \frac{1}{2} \int \left[ E_d \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) + E_d \left( \frac{\partial \mathbf{u}}{\partial t} \cdot \frac{\partial \mathbf{u}}{\partial t} \right) \right] \, dt
\]
here. This is due to the fact that the adhesive materials used in damping applications have modulus that are several orders of magnitude lower than those of the metal layers and hence, the contribution due to bending strain of the adhesive to the total energy is negligible.

The remaining effect that contributes to the potential energy is the shear in the adhesive layer. It is assumed that the shear strain varies linearly through the thickness of the adhesive. The shear strain varies from \( \gamma_b \) at the bottom of the adhesive to \( \gamma_t \) at the top. The shear strain in the adhesive can then be written as,

\[
\gamma = \gamma_b + \frac{\zeta (\gamma_t - \gamma_b)}{H_a}
\]  

where \( \zeta \) is the transverse distance from the bottom of the adhesive layer. The potential energy due to adhesive shear strain therefore equal to:

\[
V_s = \frac{1}{2} \int_{H_a}^{H_t} \left( \frac{\partial \gamma}{\partial \zeta} \right)^2 \, d\zeta
\]

where \( \gamma \) is the strain of adhesive, and \( H_a \) and \( H_t \) are the thickness of the adhesive layer.

From Figure 2 (b), it can be seen that the shear strain \( \gamma_b \) and \( \gamma_t \) may be related to the rotation and longitudinal displacement by:

\[
\gamma = -\frac{\partial}{\partial z} \left( \frac{H_a}{H_t} \right)
\]

where, as shown in Figure 1, \( H_a \) and \( H_t \) are the thickness of base beam layer and constrain layer, respectively. It should be noted that above equations are valid only for small displacements of the beams.

For the free layer damping as shown in Figure 2 (c), the kinetic energy and potential energy in the base beam layer can be expressed in the same way as for the single base beam shown in Figure 1 (a). Since the free layer is relatively soft compared to the base beam layer, we assume that the transverse displacement of the free layer be the same as the base beam layer. The free layer, however, will have its own independent longitudinal displacement \( u_f \) as shown in Figure 2 (c). The kinetic energy of the free layer can be expressed as.

\[
T_f = \frac{1}{2} \int_{H_a}^{H_t} \left[ \rho_f A_f \left( \frac{\partial u_f}{\partial t} \right)^2 + \rho_f A_f \left( \frac{\partial \theta}{\partial t} \right)^2 + \rho_f A_f \int_{H_a}^{H_t} \left( \frac{\partial u_f}{\partial \zeta} \right)^2 \frac{H_a}{H_t} \frac{\partial u_f}{\partial \zeta} \right] \, d\zeta
\]  

The contribution to the potential energy due to the extension in longitudinal direction and the bending can be expressed as,

\[
V_f = \frac{1}{2} \int_{H_a}^{H_t} \left[ E_f I_f \left( \frac{\partial u_f}{\partial t} \right)^2 + E_f I_f \left( \frac{\partial \theta}{\partial t} \right)^2 \frac{H_a}{H_t} \frac{\partial u_f}{\partial \zeta} \right] \, d\zeta
\]

The remaining effect that contributes to the potential energy is the shear in the free layer. It is assumed that the shear strain varies linearly through the thickness of the adhesive. The shear strain varies from \( \gamma_b \) at the bottom of the adhesive to zero at the top. The potential energy due to adhesive shear strain therefore equal to,

\[
V_s = \frac{1}{2} \int_{H_a}^{H_t} \frac{\partial \gamma}{\partial \zeta} \, d\zeta
\]

where, as shown in Figure 2, \( H_a \) and \( H_t \) are the thickness of base beam layer and free layer, respectively. It should be noted that all above equations are valid only for small displacements of the beams.

To solve the problem using finite element method, mass matrix and stiffness matrix for each basic component of the compound beam element need to be derived. The degree of freedom of each basic component is shown in Figure 3, and the nodal displacements of each nodal point are the same as those in Figure 2. The state vector for each basic component can be defined as,

\[
\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \theta_1 & \theta_2 & \theta_3 & \theta_4 & \ldots \end{bmatrix}^T
\]

The displacements within each basic component have to be uniquely defined by these state vectors. Following the trivial practice in finite element analysis, a set of shape functions can be defined for axial tension/compression beam element and bending beam element, respectively, as follows:
Figure 3. DOFs of basic components

And some deformation components in equation (2) through (12) can also be expressed as,

\[\begin{align*}
\theta_x &= \int \left[ \frac{1}{2} \nu \kappa_x \left( \frac{d^2 u}{dx^2} \right)^2 + \frac{1}{2} \nu \kappa_y \left( \frac{d^2 v}{dx^2} \right)^2 \right] dx \\
\theta_y &= \int \left( \frac{d^2 w}{dx^2} \right) dx \\
\end{align*}\]

Then, for the element in consideration, we have the kinetic energy and potential energy expressed in matrix format,

\[\begin{align*}
\mathbf{T} &= \frac{1}{2} \mathbf{m} \frac{d\mathbf{u}}{dt} \mathbf{u} \\
\mathbf{V} &= \frac{1}{2} \mathbf{K} \mathbf{u} \\
\end{align*}\]
For a compound beam element, its mass matrix and stiffness matrix can be assembled using those matrices corresponding to base beam layer, constrain damping layer and free layer. The nodal DOF of the compound beam element corresponding to the illustration in Figure 1 is shown in Figure 4.

![Figure 4. Nodal DOF of the compound beam element](image)

The nodal DOF can be written as,

\[
[u] = [v_{b}, v_{a}, v_{c}, v_{d}, v_{e}, v_{f}, w_{b}, w_{a}, w_{c}, w_{d}, w_{e}, w_{f}, u_{b}, u_{a}, u_{c}, u_{d}, u_{e}, u_{f}]
\]

(23)

Corresponding to the above state vector, the basic component mass matrix and stiffness matrix can be written in the sub-matrices as the following.

The base layer b as shown in Figure 4 is related to \(v_{b}\) and \(v_{b}'\), and the mass matrix and stiffness matrix can be written as,

\[
\begin{bmatrix}
[M]_b & [K]_b \\
\end{bmatrix} =
\begin{bmatrix}
M_{b} & K_{b} \\
K_{b}^T & K_{b} \\
\end{bmatrix}
\]

(24)

The adhesive layer a is related to both base layer b and constrain layer c, so the sub-matrices can be generated corresponding to \(v_{b}, v_{b}', v_{a}, v_{a}', v_{c}, v_{c}', v_{d}, v_{d}', v_{e}, v_{e}', v_{f}, v_{f}'\) as follows,

\[
\begin{bmatrix}
[M]_a & [K]_a \\
\end{bmatrix} =
\begin{bmatrix}
M_{a} & K_{a} \\
K_{a}^T & K_{a} \\
\end{bmatrix}
\]

(25)

The constrain layer c will be similar with the base layer and both matrices can be written as,

\[
\begin{bmatrix}
[M]_c & [K]_c \\
\end{bmatrix} =
\begin{bmatrix}
M_{c} & K_{c} \\
K_{c}^T & K_{c} \\
\end{bmatrix}
\]

(26)

And for free layer f, we have

\[
\begin{bmatrix}
[M]_f & [K]_f \\
\end{bmatrix} =
\begin{bmatrix}
M_{f} & K_{f} \\
K_{f}^T & K_{f} \\
\end{bmatrix}
\]

(27)

The mass matrix and stiffness matrix of the compound beam element in Figure 4 can be assembled in the same way as global mass matrix and stiffness matrix as follows,

\[
\begin{bmatrix}
[M] & [K] \\
\end{bmatrix} =
\begin{bmatrix}
M_{b} & K_{b} & M_{a} & K_{a} & M_{c} & K_{c} & M_{f} & K_{f} \\
K_{b}^T & K_{b} & K_{a}^T & K_{a} & K_{c}^T & K_{c} & K_{f}^T & K_{f} \\
\end{bmatrix}
\]

(28)

3. CASE STUDY AND COMPARISON WITH CONVENTIONAL METHOD

A five-layer-laminated beam, as shown in Fig. 5, is used in this case study to demonstrate the effectiveness of the modeling technique presented in this paper. The center layer of the laminated beam is a 0.05 mm thick soft core of adhesive with shear modulus of 1.38 x 10^5 N/m². The two constraining layers are identical and assumed to be steel of thickness of 1.5 mm with Young's modulus of 2.07 x 10^11 N/m². Two outer layers are assumed to be coating of 0.75

[402]
mm thick with Young's modulus $4 \times 10^8 \text{ N/m}^2$. Material densities are assumed to be 315.5, 7845.9, and 3074.5 kg/m$^3$, respectively. The beam length is 254 mm and assumed to be clamped at one end. The purpose of sample is to study the effectiveness of the modeling technique, so the material moduli are assumed to be constant.

Figure 5. Five-layer-laminated beam

The sample is analyzed using the compound beam element as comparison with traditional method in which a 2-D element is used to model each layer separately. As in almost all damping treatment applications, however, the adhesive layer is very thin, which results in the aspect ratio problem and requires much more elements for accurate prediction. Table 1 shows the calculated natural frequencies of the first 7 modes and the percentage error from the “accurate” value to give a convergence comparison between the compound beam element and the 2-D element on the total number of degree-of-freedom (DOF) needed by each method to achieve certain accuracy. In the compound beam element, each nodal point has 8 COFs, while 12 DOFs are needed in 2-D element. The natural frequencies calculated in Table 1 are based on the assumption that the damping material loss factor be zero for easy comparison. The natural frequency value is to be considered accurate when no significant change is observed with the increase of the number of DOF. It can be found that it only need 160 DOFs for the compound beam element to achieve the same or better convergence as 1200 DOFs for the 2-D element, which means the dramatic reduction on the computational time for eigen analysis.

To illustrate the advantage of the compound beam element over the strain energy method in modal damping calculation, comparison are given in Table 2, in which the natural frequencies ($\omega$) and modal loss factors ($\xi$) are calculated using both compound beam element and traditional 2-D element for material loss factor of $\eta=0.125$, $\eta=0.5$ and $\eta=2.0$. In traditional finite element analysis utilizing strain energy method, the natural frequency of each mode is calculated from the undamped structural, so natural frequencies are all the same no matter what is the material loss factor. In the compound beam element, the natural frequencies increase with the increase of material loss factor. This is true because when material loss factor increase, the imaginary part of the modulus increases and it should contribute to the stiffness of the beam so that the natural frequencies should increase correspondingly. Also when the material loss factor increase, the errors of modal loss factors calculated using strain energy method increase. This is because the strain energy method assumes that the damped structure has the same modal shapes as the undamped structure, which is not true and the strain energy method will not be able to give an accurate estimation to the modal loss factor.

4. CONCLUSIONS

Accurate dynamic characterization of the damped structures has been a difficult task, especially in the finite element analysis. In the present paper, a compound beam element has been developed to model all layers of a multi-layer beam into a single element. The use of the element results in significant simplification of laminated beam modeling and dramatic reduction of element density while maintaining the desired accuracy. The modeling technique allows direct consideration of complex modulus of the adhesive material in the analysis, an accurate estimation of structural modal loss factors can be obtained. It is also found that modal strain energy method can not give accurate modal loss factor estimation when material loss factor is high.

REFERENCES


[5]. ANSYS Theory Manual for Revision 5.5


### Table 1 Convergence Analysis of Laminated Beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>DOFs used in compound beam element</th>
<th>DOFs used in traditional 2-D element</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>80</td>
<td>160</td>
</tr>
<tr>
<td></td>
<td>(-0.04%)</td>
<td>(-0.01%)</td>
</tr>
<tr>
<td>2</td>
<td>120.72</td>
<td>120.59</td>
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<tr>
<td></td>
<td>(-0.2%)</td>
<td>(-0.2%)</td>
</tr>
<tr>
<td>3</td>
<td>320.15</td>
<td>319.24</td>
</tr>
<tr>
<td></td>
<td>(-0.3%)</td>
<td>(-0.2%)</td>
</tr>
<tr>
<td>4</td>
<td>619.15</td>
<td>615.22</td>
</tr>
<tr>
<td></td>
<td>(-0.6%)</td>
<td>(-0.4%)</td>
</tr>
<tr>
<td>5</td>
<td>1023.0</td>
<td>1010.4</td>
</tr>
<tr>
<td></td>
<td>(1.3%)</td>
<td>(0.7%)</td>
</tr>
<tr>
<td>6</td>
<td>1538.4</td>
<td>1504.5</td>
</tr>
<tr>
<td></td>
<td>(2.3%)</td>
<td>(0.2%)</td>
</tr>
<tr>
<td>7</td>
<td>2177.5</td>
<td>2098.2</td>
</tr>
<tr>
<td></td>
<td>(3.9%)</td>
<td>(2.0%)</td>
</tr>
</tbody>
</table>

Note: Values in ( ) indicate error percentage toward converged solution.

### Table 2 Natural Frequency and Modal Loss Factor Comparison for Laminated Beam

<table>
<thead>
<tr>
<th>Mode</th>
<th>Using compound beam element</th>
<th>Using Traditional 2-D element</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.125</td>
<td>0.5</td>
</tr>
<tr>
<td>1</td>
<td>24.157</td>
<td>0.0395</td>
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<tr>
<td>2</td>
<td>120.60</td>
<td>0.0176</td>
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<td>0.0081</td>
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<td>1502.8</td>
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<tr>
<td>7</td>
<td>2094.2</td>
<td>0.0016</td>
</tr>
</tbody>
</table>

Note: Values in ( ) indicate the percentage of inaccuracy of traditional method.