IDENTIFICATION OF TIME-VARYING MODAL PARAMETERS USING A TIME-VARYING AUTOREGRESSIVE APPROACH

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ABSTRACT:
A time-varying autoregressive model with time-varying coefficients is introduced in this paper for parameter extraction from non-stationary vibration signals. With this model, the relationship between linear time-varying modal parameters, i.e., instantaneous frequencies and damping factors, and time-varying autoregressive model coefficients is established. The time-varying autoregressive modeling is employed in order to obtain good time and frequency resolution of the time-frequency distribution of non-stationary responses. A new algorithm is presented by combining the use of base functions with the time-frequency transform of signals, in which the base functions have shift the time-varying modeling to a linear time-invariant identification problem. A simulated linear time-varying system of three degrees of freedom with time-varying stiffness is presented to validate the proposed time-varying autoregressive method, and the simulated results have demonstrated that the method based on time-varying autoregressive modeling is effective in identifying time-varying modal parameters.

NOMENCLATURE

- $x_n$: time series to be analysed
- $p$: order of the AR model
- $a_{p,l}(n)$: coefficient of the AR model
- $g_j(n)$: base functions used for expansion
- $X_n$: vector composed of $x_n g_j(n)$
- $z_r, z_r^*$: complex conjugated radixes
- $\omega_r, \xi_r$: natural frequency and damping factor
- $m$: degree of base function

1. INTRODUCTION

Modal parameters including natural frequency and modal damping ratios have been widely used to characterize linear time-invariant (LTI) mechanical systems. Modal parameters identification of structures is one of the goals of experimental studies on structural vibration. Different approaches have been proposed for the identification of LTI systems in time or frequency domain [1]. For linear time-varying (LTV) systems, Zadel [2] defined the time-varying transfer function by extending the Laplace transform to the varying impulsive response. Reference [3] used ‘varying natural frequency’ to describe the global properties of LTV systems. The concept of pseudo-modal parameters was introduced to describe dynamic properties of LTV mechanical systems [4].

LTV systems have been frequently used to model systems that have non-stationary properties and undergo small magnitude vibrations. The identification of LTV systems has received increasing attention. Many attempts have been made to address identification of LTV systems. In reference [4], the discrete-time state subspace model of freely vibrating
system is used as an identification model. It is the extension of conventional subspace technique. The feature of this method is the use of a series of input and putout data from multiple experiments.

Owing to time-varying physical parameters, vibration responses usually exhibit time-varying characteristics. Time-frequency distribution (TFD) is the most effective tool for the analysis of this kind of signals. Among all the TFDs, widely used is the so called spectrogram which comes from Short Time Fourier Transform [5]. Spectrogram has the advantage of fast computation, but it cannot give a high resolution of time and frequency simultaneously. To increase the frequency resolution, a long period of time is required. Therefore, the assumed local stationarity may not hold any more. In this circumstance, spectral components occurring in a long period may be smeared in time domain and consequently have low time resolution. To partly solve this problem, time-varying autoregressive (AR) modeling has been used as an alternative technique. In fact, frequency resolution can be enhanced because model-based methods implicitly extrapolate samples outside the windowed time period. There are many applications of time varying AR modeling in the fields of signal processing and control engineering [8-11]. However, this method has not been widely used in establishing structural dynamics. Recently, ARMA-based methods have been used in modal parameter identification of time invariant mechanical systems. In this paper, the time-varying AR modeling is introduced to identify time-varying modal parameters. The AR approach implies that a given signal is modeled as the output of a linear filter driven by a white noise process. So, the problem of structural modal parameter identification is shifted to identifying the AR model of vibration signals, the output of a linear time-varying system under white noise excitation. In particular, the time-varying coefficients of an AR process should be estimated.

In Section 2, the non-stationary AR modeling and the time-varying AR algorithm are presented. The time-varying AR spectral as well as the relation between AR coefficients and time-varying modal parameters are given. Section 3 gives simulation studies, and finally in Section 4, conclusions are summarized.

2. NON-STATIONARY AR MODELING

AR modeling takes vibration responses as the output of a linear filter driven by a white noise process. This filter referred to as AR combines the previous samples (regressive) and the output itself (auto) in a linear form [8-11]. A classical AR complex process, in a non-stationary form, can be given as:

\[ x_n + a_{p,1}(n-1)x_{n-1} + \cdots + a_{p,p}(n-p)x_{n-p} = w_n \]  

where \( \{a_{p,i}(n-i) | i = 1, \cdots, p \} \) are the time-varying AR coefficients, \( w_n \) is the Gaussian white noise and \( n \) is the sampling time.

There are two AR algorithms that can be used in the analysis of non-stationary vibration signals. First, when weak nonstationarity exists, the algorithm called adaptive AR modeling can be applied, and secondly, when strong nonstationarity exists, we can use the so called time-varying AR algorithm which estimates coefficients of base functions.

2.1. TIME-VARYING AUTOREGRESSIVE ALGORITHM

As shown in [9,10], the time-varying AR modeling of non-stationary time series is more effective. Suppose the AR coefficients are linear combinations of a set of deterministic base functions \( g_j(n) \), then \( \{a_{p,i}(n)\} \) at time \( n \) can be expressed as:

\[ a_{p,i}(n) = \sum_{j=0}^{m} a_{p,i,j} g_j(n) \]  

where \( m \) is the degree of base functions, and \( \{a_{p,i,j}\} \) are expansion coefficients. Several types of basis functions can be employed. The standard choices are the set of the Fourier base [10], Legendre polynomials [11] and the prolate
spheroidal functions [12]. Based on the characteristics of nonstationary signal to choose the basis functions.

Given equation (2), the time-varying AR model can be expressed as:

\[ a_{p,i}(n-i)x_{n-i} = \left( \sum_{j=0}^{m} a_{p,j,j} g_{j}(n-i) \right) x_{n-i} = \sum_{j=0}^{m} a_{p,j,j} g_{j}(n-i) x_{n-i} \]  

(3)

If the vector

\[ X_n = [g_0(n)x_n, \cdots, g_m(n)x_n]^T \]  

(4)

is introduced, equation (3) can be rewritten as:

\[ a_{p,i}(n-i)x_{n-i} = [a_{p,1,0} \cdots a_{p,1,m}, \cdots, a_{p,p,0} \cdots a_{p,p,m}] X_{n-i} \]  

(5)

Therefore, the AR model of time series \( \{x_n\} \) is given as:

\[ x_n + [X_{n-1}^T, \cdots, X_{n-p}^T] \theta = w_n \]  

(6)

where \( \theta = [a_{p,1,0} \cdots a_{p,1,m}, \cdots, a_{p,p,0} \cdots a_{p,p,m}]^T \).

Equation (6) represents the regression of the non-stationary process \( x_n \) over \( p \) past samples of vector \( X_n \). The parameterization shifts a linear non-stationary problem to a linear time invariant one because of the replacement of a scalar process with a vector one. The number of unknowns is multiplied by \( (m+1) \), but the cost is relatively small when compared to the benefit of keeping the problem linear.

### 2.2. TIME-VARYING PARAMETER ESTIMATION

In equation (6), \( w_n \) is taken as the residual prediction error.

Consequently, \( \theta \) is chosen as the vector that minimizes the variance of \( w_n \). In order to obtain \( \theta \), optimization should be taken, and the optimized \( \theta \) is the solution of equation (7) (Yule - Walker).

\[ A \theta = \alpha \]  

(7)

where

\[ \alpha = - \sum_{n=1}^{N} \begin{bmatrix} X_{n-1}^T \\ \vdots \\ X_{n-p}^T \end{bmatrix}, \quad A = \sum_{n=1}^{N} \begin{bmatrix} X_{n-1}^T \\ \vdots \\ X_{n-p}^T \end{bmatrix} (X_{n-1}^T \cdots X_{n-p}^T). \]

It follows that

\[ \theta = A^{-1} \alpha \]  

(8)

To reduce the cost of computation, the well-known recursive least-squares algorithm can be applied [7].

\[ \hat{\theta}_N = \hat{\theta}_{N-1} + K_N (x_N + Y_{N-1}^T \theta_{N-1}) \]

\[ K_N = -P_{N-1} Y_{N-1} Q_{N-1}^{-1} \]

\[ Q_{N-1} = 1 + Y_{N-1}^T P_{N-1} Y_{N-1} \]

\[ P_N = P_{N-1} - P_{N-1} Y_{N-1} Q_{N-1}^{-1} Y_{N-1}^T P_{N-1} \]

where vector \( Y_N \) is defined as

\[ Y_N^T = [X_{N-1}^T, X_{N-2}^T, \cdots, X_{N-p}^T] \]  

(10)

and \( P_N \) is a \([p \times (m+1)] \) by \([p \times (m+1)] \) matrix given by

\[ P_N = \left( \sum_{n=1}^{N} Y_{n-1} Y_{n-1}^T \right)^{-1} \]  

(11)

In addition, The basis dimensions are chosen by computing the criterion \( C \) defined by [8]

\[ C = 10 \log \left( \sum_{n=1}^{N} x_n^2 / \sum_{n=1}^{N} e_n^2 \right) \]  

(12)

where \( e_n \) is the residual obtained by inverse filtering of signal \( x_n \).

AR coefficients in an AR model contain the information of natural frequencies and damping factors. Because the AR model is an all-poles model, the time-varying transfer
function related to spectral estimation at time $t$ is

$$A(Z, t) = 1 + \sum_{i=1}^{p} a_{p,i}(t) Z^{-i}$$

(13)

Time-varying AR spectrum can be given as:

$$S(\omega, t) = \frac{1}{\sqrt{|A(Z, t)|^2}}$$

(14)

where $Z = e^{j\omega}$. The characteristic roots corresponding to equation (13) are related to natural frequencies and damping factors. The relation is

$$(Z_r, Z_r^*) = e^{(-\xi_r + j\sqrt{1-\xi_r^2})\omega\Delta}$$

(15)

Modal frequency and damping factor at time $t$ can be given as:

$$\omega_r = \frac{1}{\Delta} \sqrt{\ln Z_r \ln Z_r^*}$$

$$\xi_r = -\frac{\ln(Z_r Z_r^*)}{2\sqrt{\ln Z_r \ln Z_r^*}}$$

(16)

where $Z_r, Z_r^*$ complex conjugated radixes of transfer function, $\Delta$ sampling time.

3. SIMULATION STUDY OF THE PROPOSED METHOD

In this section, we carry out the identification of a 3DOF linear system with time-varying stiffness. The time-varying AR approach is applied to the output of the system under white noise excitation. The estimation of modal frequencies and damping factors is implemented via estimation of AR model coefficients. Model order selection is based on the identification of independent components in the time frequency distribution.

3.1 SIMULATION OF A 3DOF LINEAR TIME-VARYING SYSTEM

In order to validate the time-varying AR modeling for identifying time-varying modal parameters, a 3DOF linear system with time-varying stiffness is adopted. The mass, damping and stiffness matrices are as follows:

$$M = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 1000 \end{bmatrix}$$

$$C = \begin{bmatrix} 2.8 + e3 & -1.4 + e3 & 0 \\ -1.4 + e3 & 2.8 + e3 & -1.4 + e3 \\ 0 & -1.4 + e3 & 1.4 + e3 \end{bmatrix}$$

$$K = \begin{bmatrix} 1.96 + e7 & -0.98 + e7 & 0 \\ -0.98 + e7 & 1.96 + e7 & -0.98 + e7 \\ 0 & -0.98 + e7 & 0.98 + e7 \end{bmatrix} \times (1 + e^{-0.08t})$$

As indicated, stiffness varies with time. The units are respectively kg, Ns/m and N/m. Acceleration response of the system is given in figure 1. In order to check the sensitivity to measurement noise one percent Gaussian noise signal is added into the response signal.

![Figure 1. Acceleration response](image1.png)

![Figure 2. Time-frequency distribution of the](image2.png)
3.2. IDENTIFICATION OF MODAL FREQUENCIES BY TIME-VARYING AR MODELLING

Figure 2 gives the time-frequency distribution of the acceleration response, which is obtained from Gabor transform. Hanning window is adopted and its length is 128 points. From figure 2, we know that there are three components in the response. This prompts us to choose a sixth order AR model. Legendre polynomials base functions with $m=3$ are applied. Figure 3 gives the theoretical and identified results. The theoretical values of time-varying mode parameters are estimated by the parameter freezing technique \[^{[10]}\] in which the parameters are assumed to be constant in a short time interval.

![Figure 3 Theoretical modal frequencies and Identified modal frequencies](image)

4. CONCLUSIONS

Nonstationary vibration responses of a linear time-varying system can be described by time-varying AR model. In this paper we have presented a parametric method that allows us to estimate modal frequencies. The time-varying autoregressive algorithm has been applied to the estimation of AR model coefficients. Regarding model order selection, this study has combined the time-varying autoregressive method and TFD which is based on Gabor transform. In the simulation study, identified results indicate that the time-varying AR modeling is an effective tool in parameter estimation of linear time-varying systems.

REFERENCES