Application of Gabor expansion for Order Analysis

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Abstract
Automotive and machinery reliability engineers rely heavily on order analysis for examining rotating machinery. While many different techniques for order analysis have been developed, this article introduces the Gabor expansion-based order tracking approach. Compared to other known order analysis methods, this technique is not only more intuitive and more powerful, but can also be used for applications where rotational speed information is not available.

1. Introduction

One of the most popular analysis methods for engineers/scientists has been harmonic analysis. Here, the term harmonic refers to frequencies that are integer (or fractional) multiples of a fundamental frequency. In the automobile industry, such harmonics are traditionally referred to as orders. Accordingly, the harmonic analysis is called as order analysis. By analyzing the amplitudes and phases of different orders, engineers can often determine whether or not an engine is running normally.

The primary tool of conventional order analysis is the Fourier transform that assumes that the fundamental frequency does not change over time. Hence, it is only suitable for an engine running at a constant speed (see Figure 1). During engine run up/down, the frequency bandwidth of the engine sound becomes wide [10]. Consequently, the harmonics overlap. In this case, from the conventional power spectrum we are no longer able to distinguish the vibrations caused by different sources (see the plot on the left in Figure 2). On the other hand, the hidden fault is usually easier to discover at engine run up/down than at a constant speed. Hence, the technique of analyzing time-varying harmonics is very important for engine diagnostics.

One approach to track orders during engine run up/down is to apply an adaptive filter, such as the Kalman filter [13], using the engine rotation speed as a pilot to track an order in which we are interested. As an alternative, we will introduce a discrete Gabor expansion based order tracking method. Compared to the adaptive filter approach, the Gabor expansion approach is not only more robust and easier to implement, but also offers more insight into the physical process.

2. STFT Spectrogram and Discrete Gabor Expansion

One of the most common methods of studying time-varying harmonics has been the short-time Fourier transform (STFT). Unlike the traditional FFT based approaches that describe harmonics in either time or frequency domain separately, the STFT characterizes the magnitude and phase of each individual time-varying harmonic (or order) in time and frequency domain simultaneously. Although STFT offers insight into the physical process, in general it is not invertible. In some applications, however, engineers do need time waveforms of particular orders. Having corresponding time waveforms, engineers can further perform various analysis, such as cross-correlation. The process to recover the time waveform of a particular harmonic (or order) is known as the order tracking in the automobile industry. In what follows, we shall briefly introduce the basic technique of performing order tracking, that is, the Gabor transform (a modified STFT) and its inverse – the Gabor expansion.
Over the years, many different implementation schemes for discrete Gabor transform and Gabor expansion have been proposed (see [1], [6], [7], [9], [11], [12], [14], and [16]). The one presented in this paper was the extension of the method originally developed by Wexler and Raz [14]. In this method, lengths of the analysis and synthesis window functions are the same, while perfect reconstruction is guaranteed. This is a very useful property for DSP implementation.

![Fourier spectrum and Gabor coefficients](image)

Figure 1. Constant RPM (When the rotational speed of the electrical motor is constant, from the conventional FFT-based power spectrum, we can clearly identify the orders caused by coils and blades)
Figure 2. Electrical motor run up (While the orders are overlapped in the conventional FFT-based power spectrum, they can be clearly identified in the joint time-frequency domain).

For a given set of data samples $s[k]$, the corresponding discrete Gabor transform are computed by modified STFT, i.e.,

$$c_{m,n} = \sum_{k=0}^{L_s-1} s[k] \gamma^*[k - m\Delta M] e^{-j\frac{2\pi nk}{N}} \quad 0 \leq m < M \quad 0 \leq n < N$$  \hspace{1cm} (1)

where $c_{m,n}$ is also known as the Gabor coefficient. The parameters, $\Delta M$ and $N$, denote the discrete time sampling interval and the total number of frequency bins, respectively. From the filter bank point of view, $N$ is nothing more than the number of frequency bands. $\Delta M$ indicates the decimation factor. $L$ denotes the period of the sequence $s[k]$ given by,

$$\tilde{s}[k + iL] = \begin{cases} s[k] & 0 \leq k < L, \\ 0 & L_0 \leq k < L \\ i = 0, \pm 1, \pm 2, \ldots \end{cases}$$  \hspace{1cm} (2)

where $L_s$ denotes the length of the signal $s[k]$. Note that $L$ is the smallest integer that is bigger than or equal to $L_s$. Moreover, it must be evenly divided by the time sampling interval $\Delta M$ and the total number of frequency bins $N$. The integer $M$ in (1) indicates the total number of time points, that is, $M = L/\Delta M$. Accordingly, the total number of Gabor coefficients is equal to $MN$.

The ratio $N/\Delta M$ determines the Gabor sampling rate. For numerical stability, the Gabor sampling rate must be greater than or equal to one. It is named as critical sampling when $N$ is equal to $\Delta M$. For $N/\Delta M > 1$, it is named as oversampling. For oversampling, the number of Gabor coefficients $c_{m,n}$ is more than the number of original data samples $s[k]$. In this case, the transform (1), from a mathematical point of view, contains redundancy.

As long as all requirements mentioned above are satisfied, we can recover the original data samples by

$$s[k] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} c_{m,n} h[k - m\Delta M] e^{-j\frac{2\pi nk}{N}}$$  \hspace{1cm} (3)

which is known as the Gabor expansion that was first proposed by Dennis Gabor, a Nobel laureate, more than half a century ago [5].

Note that the positions of the window functions $h[k]$ and $\gamma[k]$ are interchangeable. In other words, either of them can be used as the synthesis or analysis window function. Therefore, we usually refer to $h[k]$ and $\gamma[k]$ as dual functions.

For a given window $h[k]$ (that always has a unit norm), we can compute the corresponding dual window function from $\Delta M$ independent linear systems, i.e.,

$$\sum_{q=0}^{N-1} \tilde{a}[k + p\Delta M + qN] y^*[k + p\Delta M] = \frac{\delta[q]}{N}, \quad 0 \leq k < \Delta M$$  \hspace{1cm} (4)

where $L_w$ denotes the window length. $\tilde{a}[k]$ denotes a periodic auxiliary function given by

$$\tilde{a}[k + i(2L_w - N)] = \begin{cases} h[k] & 0 \leq k < L_w \\ 0 & L_0 \leq k < 2L_w - N \\ i = 0, \pm 1, \pm 2 \end{cases}$$

If the window length is equal to the signal length, then the periodic auxiliary function $\tilde{a}[k]$ is simply
Note that the solution of (4) is not unique for oversampling. A particular interesting solution of (4) is LSE (least square error). It can be shown that for the LSE solution, the Euclidean distance between the dual functions is minimum [9], i.e.,

$$\min_{\gamma} \| \tilde{\gamma} - \tilde{h} \|^2$$

where $A\gamma = \mu$ represents the matrix form of (4). When the error is small, that is, $\tilde{\gamma} \approx \tilde{h}$, Equation (1) becomes

$$c_{m,n} = \sum_{k=0}^{M-1} \tilde{s}[k] h^*[k - m\Delta M] e^{j2\pi n k / M} \quad 0 \leq m < M \quad 0 \leq n < N$$

(5)

Then, (5) and (3) form an orthogonal-like Gabor transform pair [9]. In this case, the Gabor coefficients, $c_{m,n}$, are the signal’s projection on the synthesis window function $h[k]$. As mentioned earlier, oversampling will introduce redundancy. On the other hand, however, such redundancy leaves freedom for the selection of better window functions, $h[k]$ and $\gamma[k]$.

The Gabor expansion is a mathematical tool for characterizing a signal in the time and frequency domains jointly [4]. Although it was initially introduced in mid 40s, the relationship between STFT and the Gabor expansion was not established until the early 80s [1].

Long before [4] was developed [10], Protnoff had derived the relationship of the dual window functions [8]. However, the formula developed in [8] involves an infinite summation that is difficult, if not impossible, to process. Moreover, applying the Zak transform ([2] and [16]), we can convert the discrete Gabor expansion (1) and (3) into the form of products (similar to the convolution theorem that converts a time domain convolution into a frequency domain multiplication). The main advantage of this method is the computation speed. However, this is only the case when the lengths of the window and the signal are close to each other. When the window length is much shorter than that of the signal, or vice versa, the improvement in speed will be diminished due to huge amount of zero padding involved.

3. Discrete Gabor Expansion based Time-Varying Filter

At first glance, the pair of the discrete Gabor transform and discrete Gabor expansion seems to provide a feasible vehicle for converting an arbitrary signal from the time domain into the joint time-frequency domain, or vice versa. As a matter of fact, it is only true for $\Delta M = N$ (critical sampling). For oversampling (as is always the case for most applications), the Gabor coefficients are the sub-space of two-dimensional functions. Consequently, for an arbitrary two-dimensional function, there may be no corresponding time waveform. For example, let’s assume that we have a modified two-dimensional function

$$\hat{c}_{m,n} = w_{m,n} c_{m,n}$$

where $w_{m,n}$ denotes a user defined weighting function. Applying the Gabor expansion (1) yields

$$\hat{s}[k] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{c}_{m,n} h[k - mT] e^{j2\pi n k / M}$$

Then, we will find that in general

$$\sum_{k=0}^{M-1} \hat{s}[k] y^*[k - mT] e^{-j2\pi n k / M} \neq \hat{c}_{m,n}$$
In other words, the Gabor coefficients of the reconstructed time waveform $\hat{s}[k]$ are not equal to the modified Gabor coefficients $\hat{c}_{m,n}$ as one anticipates. Usually, we pursue a time waveform $\hat{s}[k]$ whose Gabor coefficients are most similar, in terms of the LSE, to the desired Gabor coefficients $\hat{c}_{m,n}$.

Without loss of generality, let’s rewrite the modified STFT (2) in the matrix form, i.e.,

$$\bar{c}_{MN} = G_{MN} \bar{s}_{L \times 1}$$

(6)

where $G$ denotes an analysis matrix. $\bar{c}$ is a vector whose $i$th element is equal to the Gabor coefficient $c_{(m,n)} = 0$. Accordingly, we can also write the Gabor expansion (3) as

$$\bar{s}_{L \times 1} = H_{L \times MN} \hat{c}_{MN} = H_{L \times MN} G_{MN \times L} \hat{s}_{L \times 1}$$

(7)

where $H$ denotes the synthesis matrix. Eq.(7) implies that

$$H_{L \times MN} G_{MN \times L} = I_{L \times L}$$

where $I$ denotes the identity matrix. Note that generally speaking

$$G_{MN \times L} H_{L \times MN} \neq I_{MN \times MN}$$

Then, the time waveform $\hat{s}[k]$ whose Gabor coefficients are the closest, in terms of the LSE, to the desired Gabor coefficients, is the solution of

$$\min \| G_{MN \times L} \hat{s}_{L \times 1} - \Phi_{MN \times MN} \hat{c}_{MN} \|^2$$

(8)

where $\Phi$ is a diagonal weighting matrix given by

$$\phi_{i,j} = \begin{cases} \frac{W_{(m,n)} = 0}{0} & i = j \\ 1 & i \neq j \end{cases}$$

It has been well understood that the solution of (8) is nothing more than the pseudo inverse of the matrix $G$, i.e.,

$$\hat{s}_{L \times 1} = (G_{MN \times L}^T G_{MN \times L})^{-1} G_{MN \times L}^T \Phi_{MN \times MN} \hat{c}_{MN}$$

(9)

Computation of (9) in general is enormous, particularly, when $L$ is larger. Because a typical number of data samples, in most order tracking applications, can be as many as one million, (9) is not practical for order tracking due to the size of the matrix $G$. However, if $\gamma[k] = h[k]$, then

$$H_{L \times MN} = G_{MN \times L}^T$$

(10)

Substituting (10) into (9) yields

$$\hat{s}_{L \times 1} = H_{L \times MN} \Phi_{MN \times MN} \hat{c}_{MN} = H_{L \times MN} \Phi_{MN \times MN} G_{MN \times L} \hat{s}_{L \times 1}$$

(11)

which says that when the analysis and synthesis window functions, $\gamma[k]$ and $h[k]$, are identical, the LSE solution of (9) can be directly obtained by performing the regular Gabor expansion (3) with respect to the desired Gabor coefficients!

To make the analysis and synthesis window functions, $\gamma[k]$ and $h[k]$, identical normally implies significant oversampling. For the sake of memory consumption and computation speed, we usually pursue the orthogonal-like Gabor transform introduced in the preceding section. In this case, we can often have a very similar pair of dual functions with a relatively low oversampling rate.
Finally, the solution of (11) holds for any weighting function $w_{m,n}$. There is no restriction to the selection of the weighting function $w_{m,n}$. If $w_{m,n}$ is limited to binary values (either one or zero), then it behaves as a mask, preserving $c_{m,n}$ when $w_{m,n} = 1$ and removing $c_{m,n}$ when $w_{m,n} = 0$. In this case, under certain conditions, we can achieve a time waveform whose Gabor coefficients are exactly inside area defined by the mask function $w_{m,n}$. Readers who are interested in this topic can consult reference [15].

4. Numerical Simulation

The testing was conducted on an off-the-shelf, 4-phase Electrical motor with a tachometer pulse output (two pulses for each revolution) by using a National Instruments data acquisition board and LabVIEW. Since the electrical motor contains four coils, we expect to observe harmonics (or orders) with frequencies at four, eight, twelve... times the rotation speed. In addition, it has seven blades. So, we will also expect to observe harmonics at seven, fourteen, twenty one... times the rotation speed. Figure 1 illustrates the results obtained while the electrical motor is running at a constant speed. The bottom plot depicts the tachometer pulses as well as the signal from an accelerometer mounted on the Electrical motor. The plot on the left shows the conventional FFT based power spectrum. In the middle is a joint time-frequency plot computed from the STFT with a 1024-point Hanning window. As expected, when the rotational speed is constant, we can clearly see a fourth and seventh orders from both the FFT-based power spectrum and the joint time-frequency plot. However, when the rotational speed of the electrical motor changes, in the FFT-based power spectrum, as shown in Figure 2, the distinct orders caused by the four coils or the seven blades no longer exist. However, looking at the joint time-frequency plot, we can still easily recognize the orders in which we are interested.

![Figure 3. Eighth Order Time Waveform (The vibration during 4 to 5 seconds is dominated by the eighth order – about 400Hz)](image)

Figures 3 and 4 depict the results of the Gabor expansion-based order tracking, which shows how the eighth and twelfth orders are extracted. In this example, the analysis function is a 2048-point Hanning window. The number of frequency bins $N$ is equal to the window length $L_w$. The oversampling rate is four (that is, 75% overlap). Consequently, the difference between the analysis and synthesis windows is negligible. For the sake of simplicity, we limit the weighting function $w_{m,n}$ to binary values which behaves
as a mask, preserving $c_{m,n}$ when $w_{m,n} = 1$ and removing $c_{m,n}$ when $w_{m,n} = 0$. Since the analysis and synthesis windows are almost identical, the Euclidean distance between the masked Gabor coefficients and that of the extracted time waveform, in terms of the mean square error (MSE), is minimum.

The center frequency of the selected order is automatically computed from the rotational speed, whereas the bandwidth of the weighting function is manually selected. From the time waveforms of Figures 3 and 4, it is interesting to note that this particular data record was dominated by a resonant frequency (around 400 Hz). After obtaining the time waveforms corresponding to different orders, we can further perform a variety of analysis, such as cross-correlation, phase and magnitude analysis, etc.

Unlike the model-based order tracking methods (e.g., Kalman order tracking), in which, theoretically, there is no limitation to the order resolution, the length of the window functions and the lower bound of the rotational speed restrict the order resolution of the discrete Gabor expansion based approach. In this example, the sampling frequency is 4,000 Hz and the window length is 2048-point. Thereby, each frequency bin corresponds to 4000/2048 Hz. The lower bound of the rotation speed is about 19.5 Hz, approximately 10 frequency bins. Hence, we can comfortably distinguish between components with a resolution of 0.1 orders. The longer the window function, the better the order resolution. On the other hand, however, the longer the window function, the poorer the time resolution. Consequently, we cannot arbitrarily increase the window length for a good order resolution.

![Figure 4](image)

**Figure 4.** Twelfth Order Time Waveform (The vibration, in the vicinity of 2 second, is dominated by the twelfth order – around 400Hz)

Besides the real data samples, we also apply known synthetic linear chirp signals to investigate the performance of the Gabor expansion based order tracking algorithm introduced in this paper. The start frequency of the linear chirp is 0.1Hz and the end frequency is 0.4Hz (normalized frequency). The total number of samples is 1000.

Figure 5 illustrates the relationship between the oversampling rate $N/\Delta M$ and the related error $\epsilon$(dB) given by

\[ \epsilon = \frac{10 \log_{10} \frac{\| C' - C \|}{\| C \|} }{20} \]
Obviously, the related error $\varepsilon(dB)$ reflects the accuracy of Gabor expansion based order tracking algorithm. As shown in Figure 5, with the same frequency bandwidth of the mask function $w_{m,n}$, the error $\varepsilon(dB)$ is inversely proportional to the oversampling rate. In the other words, the more the oversampling, the smaller the error. For example, when oversampling rate is 4, the related error $\varepsilon(dB)$ is $-60$dB. When oversampling rate is 16, the related error $\varepsilon(dB)$ reduces to $-160$dB. This is due to the fact that for true LES solution (10), the analysis and synthesis window functions, $\gamma[k]$ and $h[k]$, must be identical. On the other hand, the similarity between the pair of dual functions $\gamma[k]$ and $h[k]$ is directly proportional to oversampling rate. The higher the oversampling, the closer the pair of dual functions $\gamma[k]$ and $h[k]$ are.

**Figure 5.** Over-sampling Rate vs. Reconstruction Error (When the frequency bandwidth of the mask function is fixed, the reconstruction error is inversely proportional to the oversampling rate)

**Figure 6.** Frequency Bandwidth of the Mask Function vs. SNR (If the frequency bandwidth of the mask function is too narrow, then too many signals will be filtered out. On the other hand, if the frequency bandwidth of the mask function is too wider, then too much noise will be introduced. In this example, the best SNR occurs when the frequency bandwidth of the mask function is around .01Hz)
The time varying filter is also tested by adding white Gaussian noise in which the signal to noise ratio (SNR) is defined as,

$$SNR = \frac{1}{K} \sum_{k=0}^{K-1} \frac{s[k]}{(\hat{s}[k] - s[k])^2}$$

where $\hat{s}[k]$ denotes the reconstructed signal. The closer the reconstructed signal to the original signal, the larger the SNR. If the reconstructed signal $\hat{s}[k]$ is equal to the original signal $s[k]$, SNR goes to infinity.

For Gaussian noise with zero mean and variance $\delta^2$,

$$SNR = \frac{1}{\delta^2} \sum_{k=0}^{K-1} s^2[k]$$

The SNR of the testing noisy linear chirp signal is 20dB. Intuitively, if the frequency bandwidth of the mask function is too narrow, then too many signals will be filtered out. On the other hand, it also cannot be too wide. The wider the frequency bandwidth, the more noise will be included. As shown in figure 6, the best SNR occurs when the frequency bandwidth of the mask function is about 0.01Hz. In this case, the improvement of SNR is more than 15dB! It has been observed that the best frequency bandwidth of the mask function is proportional to signals’ instantaneous frequency bandwidth [10]. As the bandwidth continues to increase, however, the SNR will converge to 20dB – the original SNR.

Figure 7. Left graph is spectra of a broadband random noise sequence and selected extractions by a 1kHz Vold-Kalman Filter. Right graph is spectra of the same sequence and constant 1kHz Gabor method extractions with different mask widths. HC represents harmonic confidence. DF is equals to sampling frequency divided by the length of the window.

In addition to the features mentioned earlier, such as more robust and more insight into understanding the underlying physical process, compared to adaptive filter based methods [13], Gabor expansion based approach has far more selective. Figure 7 presents the result of an experiment for visualizing the Vold-Kalman filter shape in comparison to that in Gabor expansion based approach. To accomplish the comparison, a long sequence of uniformly distributed random noise was passed through a constant frequency Vold-Kalman filter utilizing a series of harmonic confidence factors to sharpen the filter. Note
that the bigger the harmonic confidence, the more the computational time. The spectra of these filtered sequences are illustrated at the top of the Figure 7. Next, the identical data was subjected to a Gabor expansion based order extraction by using two different mask widths both centered on 1kHz. The spectra of these sequences are shown in the lower half of Figure 7. It is obvious from these illustrations that the Gabor expansion based approach is far more selective than even the sharpest of the Vold-Kalman filters. Furthermore, in the insets, the spectra of the extracted signal components are compared more closely to the spectrum of the original signal. An inspection shows that the Gabor expansion based order extractions reproduce the spectrum of the original signal exactly inside the mask, while the Vold-Kalman filters closely approximate it, but do not reproduce it exactly. The reader can find a more comprehensive comparison study from [18].

5. Conclusion

In this paper, we introduced a discrete Gabor expansion for order tracking – a popular application in the automobile industry as well as in many other areas in which harmonics are driven by a single rotating component. Compared to other order tracking schemes, we feel the method presented in this paper is more robust and gives more insight into understanding the underlying physical process. Particularly, it is suitable for interactive analysis. As a result, one can perform order tracking without a tachometer signal, that is, drawing the mask by hand as long as the order is visible and sufficiently resolved. Although a more comprehensive comparison study is certainly needed, there is no doubt that the Gabor expansion based approach is a good alternative for the currently used techniques.

References:


