MODEL REDUCTION
AND MODEL EXPANSION
AND THEIR APPLICATIONS
PART 1 - THEORY

Peter Avitable
Modal Analysis and Controls Laboratory
University of Massachusetts Lowell
Lowell, Massachusetts

ABSTRACT
Model reduction and model expansion play an important part in many aspects of structural dynamic modeling techniques. These include reduced models for structural response studies, correlation of analytical and experimental models, expansion of measured mode shapes for structural dynamic modification and system models, and expansion for analytical model improvement studies using modal vectors. A mapping is needed between the analytical and experimental models typically used in these studies that is provided by the model reduction and model expansion processes. Contributions to this important area made by John O’Callahan are presented in this paper. This paper is broken down into Part 1 covering the theoretical aspects and Part 2 which addresses some of the applications of this material.

INTRODUCTION
In any significant effort related to structural dynamic modeling, analytical and experimental models are always employed. The analytical models are critical to the early design process as well as the further prediction of model characteristics including dynamic stress/strain, response analysis, fatigue and failure of critical components. However, these analytical models are based upon assumptions as to the actual structural characteristics and, as such, need to be validated by experimental results. These experimental models are critical to the success of any structural dynamic analysis and contain elements that cannot be obtained analytically.

However, the biggest hurdle to overcome in any use of analytical and experimental representations of a structural dynamic system is the severe mismatch of the available numbers of degrees of freedom as shown in Figure 1. The concept of model reduction and model expansion play a significant role in this important aspect of modeling especially in the efficient comparison of the large analytical set of DOF to the relatively small set of experimental DOF. In addition, these reduction and expansion processes play a significant role in the correlation and updating of analytical models as depicted in Figure 2. Many analytical modeling optimization approaches require the measured vector to be available at the full set of finite element DOF. Likewise model updating at the set of tested DOF requires the large model to be reduced that much smaller size but without distortion of the reduced model.

Many of the related techniques and their applications are presented in this paper to summarize the significant influence of these important contributions by John O’Callahan. Each technique is briefly described and applications of reduction/expansion process importance are presented. Detailed development of the theory is beyond the scope of this paper.
The bibliography at the end of the paper provides additional reference material related to reduction and expansion as well as application areas. Part 1 (this paper) addresses the theoretical aspects whereas Part 2 addresses applications.

---

**Figure 1:** Schematic of the Reduction/Expansion Concept

---

**Figure 2:** Schematic of the Overall Model Correlation/Updating Concept
THEORY OF REDUCTION AND EXPANSION PROCESSES

The general theory of each of the reduction and expansion techniques are well beyond the space limitations of this paper. At best each of the techniques can only be summarized. The specific techniques are identified with reference to appropriate papers where additional material can be found in the bibliography at the end of this paper. Applications of this material is also plentiful and reference to appropriate material is given. These techniques are summarized in the following sections along with generic applications.

**General Reduction Techniques**

Model reduction is generally performed to reduce the size of a large analytical model to develop a more efficient model for further analytical studies, such as substructuring or forced response studies. Most reduction or condensation techniques affect the dynamic character of the resulting reduced model. Model reduction is performed for a number of reasons but the technique is used primarily as a mapping technique. A schematic of the reduction process is shown in Figure 3

![Figure 3: Schematic of Reduction Process](image)

In general, a relationship between the full set of finite element DOF and the limited set of tested DOF needs to be formed

\[ \{ x_n \} = \begin{bmatrix} x_a \\ x_d \end{bmatrix} = [T]\{ x_a \} \quad \text{or} \quad \{ x_1 \} = [T_{12}]\{ x_2 \} \]  

(1)

The 'n' subscript denotes the full set of analytical DOFs, the 'a' subscript denotes the active set of DOF (sometimes referred to as master DOF and for correlation studies referred to as test DOF) and the subscript 'd' denotes the deleted DOF (sometimes referred to as embedded or omitted DOF); the [T] transformation relates the transformation between these two sets of DOFs. (Note: that the subscript 1 and 2 correspond to 'n' and 'a' respectively)

The reduced mass and stiffness is related to the full space mass and stiffness by

\[ [M_a] = [T]^T[M_n][T] \quad \text{and} \quad [K_a] = [T]^T[K_n][T] \]  

(2)

What is most important in model reduction is that the eigenvalues and eigenvectors of the original system are preserved as accurately as possible in the reduction process. If this is not guaranteed then the reduced matrices are of questionable value. The eigensolution is then given by

\[ ([K_a] - \lambda[M_a])\{ x_a \} = \{ 0 \} \]  

(3)

Depending on the reduction scheme utilized, the eigenvalues of the reduced system will generally be greater than or at most equal to the eigenvalues of the full system (ie, the reduced system is "stiffer" than the original system).

**Guyan Condensation**

In terms of introduction to the reduction process, the Guyan condensation procedure is well known. The transformation matrix relating the NDOF and the ADOF is given as
\[
\{x_n\} = \begin{bmatrix} 1 \\ -[K_{dd}]^{-1} [K_{da}] \end{bmatrix} \{x_a\} = [T_s] \{x_a\}
\] (4)

The reduced system mass and stiffness matrices are given as
\[
\begin{bmatrix} M_a^G \end{bmatrix} = [T_s]^T [M_n] [T_s] \quad \text{and} \quad \begin{bmatrix} K_a^G \end{bmatrix} = [T_s]^T [K_n] [T_s]
\] (5)

As one of the first reduction schemes, it was developed as a solution economizer and not as a mapping procedure. The inherent drawback of the Guyan reduction process is that the mass of the reduced system is not effectively preserved and therefore will generally produce reduced model frequencies that are higher than those of the original full space model.

**Improved Reduced System (IRS)**

As an extension of the Guyan reduction process, the Improved Reduced System (IRS) was developed in an attempt to account for some of the effects of the deleted DOFs that cause distortion in the Guyan reduction process. The development is based on the fact that the static structural model containing distributed forces can be condensed producing a reduced system and solution. The displacements of the reduced system are then expanded and adjusted for the deleted forces producing an exact statical solution of the complete system. A first order approximation of the eigensystem is formed using a Guyan reduced model approach which is based on the static condensation process with no adjustment for the deleted distributed inertia forces. The modal vectors of the approximate solution can be adjusted in a similar fashion as in the static solution to produce an improved set of eigenvectors. Finally, an estimate of the transformation matrix from full space to reduced space can be formed for the IRS system. The resulting transformation equation is summarized as
\[
[T_i] = \begin{bmatrix} 1 \\ [t_s] \end{bmatrix} + [t_i]
\] (6)

where
\[
[t_s] = -[K_{dd}]^{-1} [K_{da}] \quad ; \quad [t_i] = \begin{bmatrix} [0] \\ [0] \end{bmatrix} [K_{dd}]^{-1} [M_n] [T_s] [M_a]^{-1} [K_a]
\] (7)

The reduced system mass and stiffness matrices are given as
\[
\begin{bmatrix} M_a^i \end{bmatrix} = [T_i]^T [M_n] [T_i] \quad \text{and} \quad \begin{bmatrix} K_a^i \end{bmatrix} = [T_i]^T [K_n] [T_i]
\] (8)

The IRS technique generally produces a better reduced eigensystem when compared with the Guyan approach, since an estimate of the inertia associated with the deleted DOF is developed as part of the reduction process.

Many software packages have included the IRS technique into their mainstream analysis due to its significant benefits. In addition to this significant development of the IRS technique, several authors have extended the IRS technique to include additional benefits, one of which is an iterative approach that forms multiple updates of the transformation through subsequent applications of the IRS equations to converge to yet a better transformation equation.

**Dynamic Condensation**

Due to the drawbacks of Guyan condensation being only accurate as a static technique, a shifted eigen problem was formed and implemented for the development of a dynamic transformation matrix. This dynamic implementation of the Guyan reduction process is the Dynamic Condensation process which is often used in correlation studies, particularly for expansion of mode shapes.
A shift value, $f$, is introduced into the set of equations describing the dynamic system, and in a similar fashion to Guyan condensation, the dynamic transformation matrix can be written as

$$\{x_n\} = \begin{bmatrix} \mathbf{I} \\ -[\mathbf{D}_{dd}]^{-1} [\mathbf{D}_{da}] \end{bmatrix} \{x_a\} = [T_f] \{x_a\} \quad (9)$$

So the reduced mass and stiffness matrices can be written as

$$\begin{align*}
\mathbf{M}_a &= [T_f]^T [\mathbf{M}_n] [T_f] \\
\mathbf{K}_a &= [T_f]^T [\mathbf{K}_n] [T_f]
\end{align*} \quad (10)$$

Due to the formulation of the dynamic condensation process, the eigensolution of the reduced matrices will result in one eigenvalue which will correspond to the shift value used for the reduction process. If the shift value happens to correspond exactly to one of the eigenvalues of the system, then this eigenvalue will be preserved accurately in the reduced model and will also produce an expanded eigenvector which will be exactly the same as the corresponding eigenvector from the full finite element model relating to the shifted eigenvalue. None of the other eigenvalues will necessarily correspond exactly to any of the eigenvalues of the full system.

The dynamic transformation matrix is not typically used for reduction since only one eigenvalue or frequency value is preserved. However, its benefits are most significant in the expansion of mode shapes as well as the expansion of measured frequency response functions.

**System Equivalent Reduction Expansion Process (SEREP)**

The SEREP modal transformation relies on manipulation of the modal mapping equation to form its transformation using

$$\{x_n\} = \begin{bmatrix} \{x_a\} \\ \{x_d\} \end{bmatrix} = \begin{bmatrix} \mathbf{U}_a \\ \mathbf{U}_d \end{bmatrix} \{p\} \quad (11)$$

Using a generalized inverse, this can be manipulated to give

$$\{p\} = \left( [\mathbf{U}_a]^T [\mathbf{U}_a] \right)^{-1} [\mathbf{U}_a]^T \{x_a\} = [\mathbf{U}_a]^g \{x_a\} \quad (12)$$

which then is used to relate the NDOF to the ADOF as

$$\{x_n\} = [\mathbf{U}_n] [\mathbf{U}_a]^g \{x_a\} = [T_u] \{x_a\} \quad (13)$$

where

$$[T_u] = [\mathbf{U}_n] [\mathbf{U}_a]^g \quad (14)$$

This is the SEREP transformation matrix that is used for either the reduction of the finite element mass and stiffness matrices or for the expansion of the measured experimental modal vectors.

The System Equivalent Reduction Expansion Process (SEREP) relies on a finite element model or analytical model from which an eigensolution is obtained for developing the mapping between the full set of 'n' finite element DOF and the reduced set of 'a' DOF. The eigensolution of the full set of system matrices yields a set of modal vectors which can be partitioned...
into those degrees of freedom that correspond to the active set of 'a' DOF and the inactive set of 'd' DOF. This is shown pictorially in Figure 4.

Figure 4: Schematic of Ua Partition of Un

Since the transformation matrix is formed from eigenvalues and eigenvectors of the full space finite element model, the reduced matrices are exact in the sense that the vectors are preserved in an exact sense in the reduced model. This implies that any collection of desired vectors can be retained in an exact form in the reduced model. This is significant in terms of the development of efficient models from large finite element models used for forced response studies, especially those that contain discrete nonlinear effects that are typical in joints and connectors that are contained in many structural systems.

While the size of these reduced mass and stiffness matrices is 'a' by 'a', the rank of the reduced matrices is only 'm'. Therefore, use of these matrices must be done with caution. Due to this rank deficiency, an alternate form of the SEREP reduction process which invokes an exact solution can be obtained by using a=m for the reduction; this is commonly referred to as SEREPa. Due to some of these rank deficiency implications, variants of SEREP have been developed. Most notably, the Hybrid technique which combines the SEREP and GUYAN processes as

$$\begin{align*}
\end{align*}$$

There are other significant computational benefits of SEREP that are often not noticed due to the simplicity of its development. Using the SEREP transformation matrix, the reduced mass and stiffness matrices can then be written as

$$\begin{align*}
[M_a] &= [T_U]^T[M_n][T_U] \\
[K_a] &= [T_U]^T[K_n][T_U] \\
\end{align*}$$

Several important manipulations using the SEREP equations are very important to show that the reduced mass and stiffness matrices can be formed without the original mass and stiffness matrices as

$$\begin{align*}
\end{align*}$$

and has extremely significant computational implications – that is that the mass and stiffness of the original system are not needed in order to form the reduced matrices. These implications extend from efficient formulation of reduced matrices to significant implications in terms of correlation of analytical and experimental models.

The SEREP process is heavily used in commercially available software packages due to its significant numerical advantages and has been a significant computational tool for various aspects of structural dynamic modeling techniques.
Mode Shape Expansion

Expansion is another aspect of structural dynamic modeling that is closely tied to model reduction. Obviously, all of the reduction schemes are also useful for expansion processing. Further aspects of expansion are discussed here. A schematic of the expansion process is shown in Figure 5.

![Figure 5: Schematic of the Expansion Process](image)

Early expansion techniques evolved around using spline fits and polynomial expansion based on geometry and measured data. While in concept they are useful, in practice, using these approaches for general structural systems, is not feasible. Most expansion techniques utilized today, involve the use of the finite element model as a mechanism to complete the unmeasured DOF from the experimental modal model. In essence, the finite element model is used as a high order polynomial curvefitter to estimate the experimental mode shapes at the deleted DOF. The majority of the expansion techniques use the model reduction transformation matrix as an expansion mechanism. The basic relationship relating the ADOF to the NDOF is given by (1) and is used to expand measured mode shapes using

$$[E_n] = [E_a] = [T][E_a]$$

(18)

The measured experimental modal vectors at ADOF are expanded over all the finite element NDOF using the transformation matrix $[T]$. This transformation matrix will take on various forms depending on which technique is utilized.

Of course, the expansions of Guyan using (4), IRS using (6), Dynamic condensation using (9) and SEREP using (13) are all important techniques that are used for the expansion process.

### Guyan Expansion

$$[E_a] = [T_i][E_a] = \left[ \begin{array} {c} \mathbf{I} \\ -[K_{dd}]^{-1}[K_{da}] \end{array} \right][E_a]$$

(19)

### IRS Expansion

$$[E_a] = [T_i][E_a] = \left[ \begin{array} {c} \mathbf{I} \\ -[K_{dd}]^{-1}[K_{da}] \end{array} \right][E_a]$$

(20)

### Dynamic Expansion

$$[E_a] = [T_i][E_a] = \left[ \begin{array} {c} \mathbf{I} \\ -[D_{dd}]^{-1}[D_{da}] \end{array} \right][E_a]$$

(21)

### SEREP Expansion

$$[E_n] = [T_u][E_a] = [U_n][U_a]^{\mathbb{G}}[E_a] = \left[ \begin{array} {c} [U_n] \\ [U_d] \end{array} \right][U_a]^{\mathbb{G}}[E_a]$$

(22)

In general, the same comments that applicable for model reduction also apply in regards to expansion. The Guyan process is always sensitive to the selection of ADOF and will generally require many DOF in order to get a good approximation of the
expanded mode shape since mass is not included in the process. The IRS technique will improve on the Guyan process but will never provide exact results even when iterative IRS is employed. The Dynamic expansion will provide an excellent expanded mode shape for at most one mode of the system. The SEREP is the only one of the techniques that will provide an exact mapping of the NDOF to ADOF information (as implied by its name as the System Equivalent Reduction and Expansion Process). Several important features of the expansion process need to be pointed out.

It is extremely important to notice that Guyan, IRS and Dynamic expansion all contain an [I] in the upper partition of the expansion transformation. This implies that the measured ADOF mode shape is preserved in its original form from the experiment. However, the SEREP process does not enforce this restriction unless so desired by the user in the expansion process. Notice that the ADOF may be changed as seen by the upper partition of this equation

\[
\left[ E_a \right]' = [U_a][U_a]^T\left[ E_a \right]
\]

and that the deleted DOF are estimated by

\[
\left[ E_d \right] = [U_d][U_a]^T\left[ E_a \right]
\]

This very important aspect of SEREP is generally referred to as completion (24) and smoothing (23). While it may seem that the measured data should not be affected by the expansion process, generally, the data contains imperfections and the smoothing process is an important consideration of this least squares process of estimating expanded shapes.

**SOME APPLICATIONS OF REDUCTION AND EXPANSION PROCESSES**

There are many areas of application of reduction and expansion. The model reduction and expansion process has played an important part in the development of the Pseudo Orthogonality Check for correlating analytical and experimental modal vectors. In addition, it was extended to form the Coordinate Orthogonality Check which is a very useful mass scaled degree of freedom correlation tool. The actual beginnings of expansion were necessary in the early days of structural dynamic modification where realistic elements could not be used due to the lack of rotational DOF. Beyond vector expansion, the expansion processes were also instrumental in the expansion of frequency response functions for estimations of rotational degrees of freedom needed for impedance system models. In addition, the formation of reduced models were extremely useful in the development of system models for forced response analysis where the components were interconnected using highly nonlinear connection elements. These are a few of the more important applications of model reduction and model expansion that are described in Part 2 of this paper.

**SUMMARY**

The important aspects of model reduction and model expansion contributions made by John O’Callahan are presented in this paper. The theory of the techniques are summarized along with commentary as to the use and application of the various techniques is this Part 1 of this paper. Part 2 presents the more important application of the model reduction and model expansion processes.
NOTATION

Matrix
- \([M]\) analytical mass matrix
- \([C]\) analytical damping matrix
- \([K]\) analytical stiffness matrix
- \([U]\) analytical modal matrix
- \([I]\) diagonal modal mass matrix
- \([Ω^2]\) diagonal modal stiffness matrix
- \([T]\) transformation matrix
- \([E]\) experimental modal vectors
- \([D]\) dynamic matrix

Vector
- \(\{x\}\) acceleration
- \(\{\dot{x}\}\) velocity
- \(\{x\}\) displacement
- \(\{F\}\) force
- \(\{p\}\) modal displacement
- \(\{u\}\) modal vector
- \(\{e_i\}\) ith experimental modal vector
- \(\{u_j\}\) jth analytical modal vector

Subscript
- \(n\) full set of finite element DOF
- \(a\) tested set of experimental DOF (also master or active DOF)
- \(d\) deleted (omitted) set of DOF
- \(S\) static condensation
- \(I\) IRS condensation
- \(f\) Dynamic condensation
- \(U\) SEREP condensation
- \(H\) Hybrid condensation
- \(k,p\) degree of freedom identifiers
  (k also used as mode identifier in some experimental modal terminology)
- \(i,j\) mode identifiers
  (also used as DOF identifiers in some experimental modal terminology)

Superscript
- \(T\) transpose
- \(g\) generalized inverse
- \(-1\) standard inverse
- \(*\) conjugate

ACRONYMS

- FEM finite element model
- EMA experimental modal analysis
- SEREP system equivalent reduction expansion process
- IRS improved reduced system
- MAC modal assurance criteria
- CoMAC coordinate modal assurance criteria
- POC pseudo orthogonality check
- CORTHOG coordinate orthogonality check
- FRF frequency response function
- DOF degree of freedom
- NDOF full set of 'n' finite element DOF
- ADOF reduced set of 'a' test DOF
- DDOF remaining set of 'd' deleted or omitted DOF
REFERENCES


17. Automatic Selection of Reduced Degrees of Freedom, J.O'Callahan, P.Li, Twelfth International Modal Analysis Conference, Honolulu, Hawaii, Feb 1994


23. SEREP Expansion , J.C.O'Callahan, P.Li, Fourteenth International Modal Analysis Conference, Detroit, MI, Feb 1996


