Non-linear Free Vibration Identification via the Wavelet Transform

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ABSTRACT

This paper presents two time methods to identify weak non-linearities on damping and stiffness of a free vibrating system. Only the displacement is known. The first method is a direct parameter estimation method: using the measured displacement data of the system we obtain by numerical differentiation velocity and acceleration. A linear system containing the parameters is then derived. A least squares and a total least squares procedures are applied to obtain the physical parameters. The second method uses the wavelet transform: using the ridges and skeletons of the wavelet transform, we identify the parameters of the non-linear vibrating system. These methods are applied to simulated single degree of freedom systems with non-linearities on damping and stiffness.

1. INTRODUCTION

The aim of this paper is to compare two different methods applied to the identification and quantification of non-linearities on damping and stiffness of a vibrating system. We consider the free vibration of the vibrating system and only the displacement is measured. We use two methods to obtain the non-linear parameters. The first method, called a direct estimation method, is based on the time histories of the displacement. Using its derivatives we obtain time histories of velocity and acceleration. A linear system containing the unknown parameters is then obtained. This system is solved by a least squares and a total least squares procedure. The second method uses the wavelet transform of the displacement. The wavelet transform is a procedure that converts a time response representation into a time-frequency response and has the advantage that it requires to measure only the displacement of the system. It is not necessary to know velocity and acceleration. The wavelet transform has been used successfully to obtain natural frequencies and damping ratios of linear vibrating systems. Numerical and experimental results are presented in [1-3]. Identification of non-linear systems using the wavelet transform has been studied by W.J. Staszewski [4] and P. Argoul and T. Le [5]. W. J. Staszewski [4] uses the Morlet wavelet as analysing function and presents two simulated examples. The first example analyses a single degree of freedom system with Coulomb friction and cubic stiffness non-linearities. The second example analyses a two degrees of freedom system with cubic stiffness non-linearities. These non-linearities are identified but not quantified. In [5], P. Argoul and T. Le consider the Cauchy wavelet as analysing function and use accelerometer responses of a clamped beam. With limitation to the first mode, they represent the beam by an oscillator with a cubic non-linearity and a weak viscous damping. In this paper we use the direct parameter estimation method and the wavelet transform to identify and quantify non-linearities of vibrating systems.

This paper is organized as follows. In section 2 the approximate equations of amplitude and phase are derived for a single degree of freedom system with non-linearities on damping and stiffness. In section 3 the direct parameter estimation method is briefly presented. In section 4 the wavelet transform and its properties are introduced. The main concept of wavelet transform ridges, skeletons and free response recovery
procedures are also described. In section 5 we apply these two methods to simulated SDOF systems with non-linearities on damping and stiffness. This paper is briefly concluded in section 6.

2. RESPONSE OF A NON-LINEAR SINGLE DEGREE OF FREEDOM SYSTEM

2.1 General case
Consider the general differential equation governing the free vibration of systems having a single degree of freedom

$$\ddot{x} + \omega_n^2 x + \varepsilon f(\dot{x}, x) = 0$$  \hspace{1cm} (1)

where \( \omega_n \) is the natural angular frequency of the linear system, \( \varepsilon \) is a small dimensionless parameter and \( f(\dot{x}, x) \) a general non-linear function of displacement \( x \) and velocity \( \dot{x} \). The dot indicates time derivative, as usual. It is well known that for the corresponding linear problem \( (\varepsilon = 0) \) the solution is \( x(t) = A \cos(\omega_n t + \beta) \) where \( A \) and \( \beta \) are constants. For the determination of the analytical solution of equation (1) we use the method of variation of parameters [6], [7] or the method of Krylov-Bogoliubof [6]. Using this method a solution to a non-linear equation (1) can be sought in the form

$$x(t) = A(t) \cos(\omega_n t + \beta(t)) = A(t) \cos(\varphi(t))$$  \hspace{1cm} (2)

where \( \varphi(t) = \omega_n t + \beta(t) \); \( A(t) \) and \( \varphi(t) \) are the amplitude and phase modulation of the system free response which are time-dependent functions. However, this procedure introduces an excessive variability into the solution; consequently, an additional restriction may be introduced: it is convenient to require the velocity \( \dot{x}(t) \) to have the same form as the harmonic oscillator. The expressions describing the variations of \( A(t) \) and \( \varphi(t) \) are obtained from [6]:

$$\dot{A}(t) = -\frac{\varepsilon}{\omega_n} K_0(A) = \frac{\varepsilon}{2\pi\omega_n} \left[ \left( A(t) \cos(\varphi(t)), -\omega_n A(t) \sin(\varphi(t)) \right) \right] d\varphi$$  \hspace{1cm} (3)

$$\dot{\beta}(t) = \left( \frac{\varepsilon}{\omega_n A(t)} \right) P_0(A) = \left( \frac{\varepsilon}{2\pi\omega_n A(t)} \right) \left[ \left( A(t) \cos(\varphi(t)), -\omega_n A(t) \sin(\varphi(t)) \right) \right] d\varphi$$  \hspace{1cm} (4)

These two equations allow to easily obtain an approximate analytical solution describing the free behaviour of a single degree of freedom system, for different forms of the non-linear function \( f(\dot{x}, x) \).

2.2 Envelope and phase of a non-linear damping system
By application of the above procedure consider a composite damping system which is defined by

$$\varepsilon f(\dot{x}) = \sum_{i=0}^{P} \mu_i |\dot{x}|^i \text{sgn}(\dot{x})$$  \hspace{1cm} (5)

where \( P \) is the order considered in the damping system and \( \mu_i \) is the \( P^i \) damping coefficient normalised to the mass. The approximate free response of a single degree freedom system with a composite damping can be obtained using (3) and (4)

$$\dot{A}(t) = -\frac{1}{\pi\omega_n} \left[ \left( \sum_{i=0}^{P} \mu_i \omega_n^i \dot{A}(t) \sin^{i+1} \varphi d\varphi \right) \right] - \sum_{i=0}^{P} \frac{1}{\sqrt{\pi}} \mu_i \omega_n^{i-1} \frac{\Gamma(i/2 + 1)}{\Gamma(i/2 + 3/2)} \left( A_i(t) \right)$$  \hspace{1cm} (6)

where \( \Gamma \) is the gamma function [7] and

$$c_i = -\frac{1}{\sqrt{\pi}} \mu_i \omega_n^{i-1} \frac{\Gamma(i/2 + 1)}{\Gamma(i/2 + 3/2)}$$  \hspace{1cm} (7)
\[ \beta(t) = \frac{1}{2\pi\omega_n A(t)l} \int_0^l 2\pi \sum_{i=0}^p \mu_i \left[ -\omega_n A(t) \sin \left( \varphi(t) \right) \right]^i \operatorname{sgn} \left( -\omega_n A(t) \sin \left( \varphi(t) \right) \right) \cos \varphi d\varphi = 0 \]  

This value implies that for a composite damping mechanism the phase angle \( \beta(t) \) does not change over time: \( \beta(t) = \beta_0 \), which is constant. It is important to note that non-linearities on damping have only an impact on the envelope \( A(t) \) and they do not affect the phase \( \beta(t) \) which remains constant.

Our objective is to determine \( A(t) \), \( \dot{A}(t) \) and use (6) for the identification of the order \( p \) and the estimation of coefficients \( c_i \). The envelope \( A(t) \) will be determined from the ridges of the wavelet transform presented in the next section. From the couple \((p, c_i)\), it is easy to identify the composite damping mechanism and to quantify the damping coefficients \( \mu_i \) from (7).

### 2.3 Envelope and phase of a non-linear damping and non-linear stiffness system

Consider a single degree of freedom system with non-linearities on damping and stiffness of the form

\[ \varepsilon f(\dot{x}, x) = \sum_{i=0}^p \mu_i \left[ x_i \right]^i \operatorname{sgn}(x) + \eta_q |x|^q \operatorname{sgn}(x) \]  

where \( \eta_q \) is the \( q \)-th order mass normalised non-linear stiffness. From (3), the expression for \( \dot{A}(t) \) is the sum of two terms \( (\dot{A}(t) = \dot{A}_1(t) + \dot{A}_2(t)) \): the term due to non-linear damping \( \dot{A}_1(t) \) and the term due to non-linear stiffness \( \dot{A}_2(t) \). It is easy to show that \( \dot{A}_2(t) = 0 \), so non-linearities on stiffness do not affect the amplitude decay of the signal; only non-linearities on damping affect the envelope of the signal. The expression for \( \beta(t) \) is again the sum of two terms \( (\beta(t) = \beta_1(t) + \beta_2(t)) \): the term due to non-linear damping \( \beta_1(t) = 0 \) and the term due to non-linear stiffness \( \beta_2(t) \) which is given by

\[ \beta_2(t) = \frac{\eta_q}{2\pi\omega_n A(t)} \int_0^l 2\pi A(t) \cos \left( \varphi(t) \right)^q \operatorname{sgn} \left( A(t) \cos \left( \varphi(t) \right) \right) \cos \varphi d\varphi = r_q A^{q-1}(t) \]  

where

\[ r_q = \frac{\eta_q}{2\omega_n} \ln \left[ 1 + (-1)^{q+1} \frac{\Gamma(q+2)}{2^{q+1} \left( \Gamma(q/2 + 3/2) \right)^2} \right] \]  

It is important to note that non-linearities on damping do not affect the phase of the signal. Only non-linearities on stiffness have an influence on the phase \( \beta(t) \) of the signal. Our objective is to determine \( \beta(t) \) and use (10) and (11) for the identification of the order \( q \) and quantification of non-linearities on stiffness from \( r_q \). The phase \( \beta(t) \) will be determined from the ridges of the wavelet transform.

### 3. DIRECT PARAMETER ESTIMATION METHOD

Only the displacement of the vibrating system is known. This displacement is sampled at regular intervals and the system data is then a discrete set of values: \{ \( x_k \) \} with \( k = 1, 2, \ldots, N \). The derivatives \( \dot{x}(t) \) and \( \ddot{x}(t) \) can be approximated by a numerical differentiation using a five points formulae:

\[ \dot{x}_k = \frac{1}{12\Delta t} (\dot{x}_{k+2} + 8\dot{x}_{k+1} - 8\dot{x}_{k-1} + \dot{x}_{k-2}) \; ; \; \ddot{x}_k = \frac{1}{12\Delta t} (-\ddot{x}_{k+2} + 8\ddot{x}_{k+1} - 8\ddot{x}_{k-1} + \ddot{x}_{k-2}) \]  

At each sampling instant we have the following equation:

\[ \ddot{x}_k + \omega_n^2 x_k + \sum_{i=0}^p \mu_i \left[ x_k \right]^i \operatorname{sgn}(x_k) + \eta_q |x_k|^q \operatorname{sgn}(x_k) = 0 \]  

\( \text{(13)} \)
Considering all the N samples of displacement, velocity and acceleration this equation can be assembled into a matrix relation

\[
\begin{bmatrix}
    x_1 & \text{sgn}(x_1) & \dot{x}_1 & \cdots & |\dot{x}_1|^p \text{sgn}(\dot{x}_1) & |\ddot{x}_1|^q \text{sgn}(\ddot{x}_1) \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    x_{N-8} & \text{sgn}(x_{N-8}) & \dot{x}_{N-8} & \cdots & |\dot{x}_{N-8}|^p \text{sgn}(\dot{x}_{N-8}) & |\ddot{x}_{N-8}|^q \text{sgn}(\ddot{x}_{N-8})
\end{bmatrix}
\begin{bmatrix}
    \omega_n^2 \\
    \mu_0 \\
    \mu_1 \\
    \vdots \\
    \mu_p \\
    \eta_q
\end{bmatrix} = \begin{bmatrix}
    -\dddot{x}_1 \\
    -\dddot{x}_2 \\
    \ddots \\
    \ddots \\
    -\dddot{x}_{N-8}
\end{bmatrix}
\] (14)

we obtain a linear system and the parameters \((\omega_n, \mu_0, \mu_1, \ldots, \mu_p, \eta_q)\) are obtained by a least squares procedure using the singular value decomposition. A total least squares procedure [10] is also applied to obtain these parameters.

4. THE CONTINUOUS WAVELET TRANSFORM

4.1 Definitions and theoretical background

The Fourier transform, though an important analysis tool, is limited by its ability to provide only the frequency information contained in the signal. A time-frequency representation of the signal is needed when the time localization of the spectral components is required and the signal possess non-stationary characteristics. Though the short time Fourier [8] was developed to overcome the limitations of the Fourier transform, the problem of selecting the window size limited its usefulness. Narrow windows gave good time resolution with poor frequency resolution and wide windows gave good frequency resolution with poor time resolution. Although time and frequency resolution problems exist regardless of the transform used, it can be avoided by an alternative approach called multiresolution analysis, or wavelet transform analysis, which uses narrow windows for the analysis of low frequencies and wide windows for high frequencies. The wavelet transform of the function \(x(t)\) is defined as [8],[9]

\[
W_{\psi} \left[ x \right](a, b) = \langle x, \psi_{a,b} \rangle = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} x(t) \psi^* \left( \frac{t - b}{a} \right) dt
\] (15)

where \(\psi(t)\) is the analyzing or mother wavelet and \(\psi^*(t)\) the complex conjugate of \(\psi(t)\); \(b\) is the parameter localizing the wavelet function in the time domain and is called translation parameter. \(W_{\psi} \left[ x \right](a, b)\) shows the local information about \(x(t)\) at the time \(t = b\). The coefficient \(a\) is the dilatation or scale parameter defining the analysing window stretching.

A very useful property of the wavelet transform is its linearity: the wavelet transform of \(P\) signals is

\[
W_{\psi} \left[ \sum_{i=1}^{P} x_i \right](a, b) = \sum_{i=1}^{P} W_{\psi} \left[ x_i \right](a, b)
\] (16)

This property is very convenient for the analysis of multi-component signals: it is possible to analyse each component \(x_i\) of a multi-component signal.

3.2 The modified Morlet wavelet

One of the most known and widely analyzing function used is the Morlet wavelet defined in the time domain as \(\psi(t) = e^{-\frac{t^2}{2}} e^{j\omega_0 t}\), where \(\omega_0\) is the wavelet frequency. The dilated version of the Fourier transform is

\[\hat{\psi}(ao) = \sqrt{2\pi} e^{-\frac{1}{2}(ao - \omega_0)^2}.\]

In practice the value of \(\omega_0\) is chosen \(\omega_0 \geq 5\). Note that \(\hat{\psi}(ao)\) is maximum at the central frequency \(\omega_c = \omega_0 / a\) and the Morlet wavelet can be viewed as a linear bandpass filter whose bandwidth is proportional to \(1/a\) or to the central frequency. Thus, the value of the dilatation parameter \(a\) at which the wavelet filter is focused on the wavelet frequency can be determined from \(a = \omega_0 / \omega_c\).

The wavelet transform analyses an arbitrary function \(x(t)\) only locally at windows defined by a wavelet function. The wavelet transform decomposes \(x(t)\) into various components at different time windows and
frequency bands. The size of the time window is controlled by the translation parameter \( b \) while the length of the frequency band is controlled by the dilatation parameter \( a \). Hence, one can examine the signal at different time windows and frequency bands by controlling translation and dilatation. However, constrained by the uncertainty principle [8], a compromise usually has to be made choosing either a narrow time window for good time resolution, or a wide time window for good frequency resolution. In order to control the shape of the analyzing wavelet we introduce the modified Morlet wavelet.

We consider the Morlet wavelet function and introduce a parameter \( N \) which controls the shape of the basic wavelet: this parameter balances the time resolution and the frequency resolution of the Morlet wavelet. The modified Morlet wavelet function used in this paper is:

\[
\psi(t) = e^{-\frac{t^2}{N}} e^{j\omega_0 t}
\]

with \( N > 0 \) and whose dilated version of its Fourier transform is

\[
\hat{\psi}(a\omega) = \sqrt{N\pi} e^{-\frac{N}{4}(a\omega-\omega_0)^2}
\]

An important value of \( N \) gives a narrower spectrum allowing a better frequency resolution, but at the expense of time resolution. So, there always exists an optimal \( N \) that has the best time-frequency resolution for a certain signal localized in the time-frequency plane. This modified Morlet wavelet function offers a better compromise in terms of localization, in both time and frequency for a signal, than the traditionally Morlet wavelet function. The optimal value of \( N \) is obtained by minimizing the entropy of the wavelet transform [2].

### 3.3 Ridge and skeleton of the wavelet transform

S. Mallat [8] and B. Torresani [9] give the definition of a class of signals called asymptotic and present some results for the time-frequency analysis of such signals. A signal of the form (2) is asymptotic if the amplitude \( A(t) \) varies slowly compared to the variations of the phase \( \phi(t) \), or in a more rigorous way

\[
\dot{\phi}(t) = \omega_0 + \dot{\beta}(t) > 0 \quad \text{and} \quad \left| \frac{1}{\frac{\phi(t)}{A(t)}} \right| < \epsilon
\]

where \( \epsilon \) is a small real positive number. The analytic signal associated with the asymptotic signal is

\[ x_a(t) = A(t)e^{j\phi(t)} \]

and from this definition, S. Mallat [8] proposed the concept of instantaneous angular frequency as the time varying derivative of the phase: \( \omega(t) = \dot{\phi}(t) \). The wavelet transform of an asymptotic signal \( x(t) \) is obtained by asymptotic techniques and is expressed as [8],[9]

\[
W_x(a,b) = \frac{\sqrt{\hat{a}}}{2} A(b)e^{j\phi(b)} \psi^*(a\phi(b))
\]

Using the modified Morlet wavelet, we have

\[
W_x(a,b) = \frac{\sqrt{\hat{a}}}{2} \sqrt{N\pi} A(b)e^{-\frac{N}{4}(a\phi(b)-\omega_0)^2} e^{j\phi(b)}
\]

The square of the modulus of the wavelet transform can be interpreted as an energy density distribution over the time-scale plane. The energy of the signal is essentially concentrated on the time-scale plane around a region called the ridge of the wavelet transform. In other words, the ridge of the wavelet transform is the region containing the points defined by \( a = a(b) \), where the amplitude of the wavelet transform is maximum. The ridges are identified by seeking out the points where wavelet coefficients take on local maximum values: for each value of \( b \), we obtain a value of \( a \) such as

\[
\left| W_x(a,b) \right| = \max_a \left| W_x(a,b) \right|
\]

To obtain the ridge, the dilatation parameter \( a = a(b) \) has to be calculated in order to maximize the analysing wavelet

\[
\psi^*(a\phi(b))
\]

that is using the modified Morlet wavelet, for \( a = a(b) = \frac{\omega_0}{\dot{\phi}(b)} \). The values of the wavelet transform that are restricted to the ridge are the skeleton of the wavelet transform, we obtain
It is important to note that the real components of the wavelet transform along the ridge are directly proportional to the signal given by (2) and from (22) we obtain

\[ A(b) = 2 \frac{|W_{\psi}[x](a(b), b)|}{\sqrt{N \pi a(b)}} \]  

(23)

\[ \phi(b) = \text{Arg} \left( W_{\psi}[x](a(b), b) \right) \]  

(24)

The ridge and the skeleton of the wavelet transform will be used for the estimation of the instantaneous amplitude \( A(t) \) and instantaneous angular frequency \( \dot{\phi}(t) \). From (23) and (24) we can compute \( \dot{A}(t) \) and \( \ddot{\phi}(t) = \dot{\phi}(t) - \omega_n \) that will be used with (6) and (10) to identify non-linearities on damping and stiffness and quantify their values.

Finally, we use the following procedure for the identification and quantification of non-linearities in vibrating systems. Once the free response of the mechanism has been measured the skeleton of the wavelet transform is extracted to obtain the envelope \( A(t) \) and its derivative \( \dot{A}(t) \). We form then the equation

\[ \dot{A}(t) = \sum_{i=0}^{P} c_i \dot{A}_i(t) \]  

If \( p = 1 \), we obtain the equation of a straight line and from the slope of this line we estimate \( c_1 \) and the damping coefficient from (7). If \( p \neq 1 \), a polynomial is obtained. The coefficients \( c_0, c_1, \ldots, c_p \) of the polynomial, of a specified degree \( p \), are those that best fit the data in a least squares sense. Once the coefficients \( c_0, c_1, \ldots, c_p \) have been computed the damping coefficients are estimated from (7). From the skeleton of the wavelet transform we extract also the phase \( \phi(t) \) and its derivative \( \dot{\phi}(t) \), representing the instantaneous angular frequency of the system. Representations of vibration behaviour in the form of curves of free vibration envelope versus instantaneous frequency are called backbone curves. For linear systems a backbone does not depend of the envelope and is constant. For systems with non-linearities on stiffness the backbone is not constant and following the form of the backbone we obtain a softening system or a hardening system. The backbone is obtained from \( \dot{\phi}(t) = \omega_n + r_q A^{q-1}(t) \) and the natural frequency \( \omega_n \) is estimated from the backbone for \( A(t) = 0 \). The degree \( q \) of the polynomial and the coefficient \( r_q \) are obtained by minimization of the normalized root mean square error (RMSE) between the measured value of the instantaneous angular frequency \( \dot{\phi}(t) \) obtained from the skeleton of the wavelet transform and the identified value \( \hat{\dot{\phi}}(t) \) of the mechanical model

\[ \text{RMSE}(\hat{\dot{\phi}}) = \sqrt{\frac{1}{T} \sum_{i=1}^{T} \left[ \dot{\phi}(t_i) - \hat{\dot{\phi}}(t_i) \right]^2} \]  

(25)

where \( T \) is the number of samples. The value of the normalized root mean square error is a measure of the accuracy of the fit. Once \( r_q \) has been obtained the stiffness coefficient is estimated from (11). For multi component signals we use the time-frequency localisation properties of the wavelet transform and the property of linearity. It is possible to follow for each \( \tilde{P} \) mode the envelope decay \( A_i(t) \), the phase variations \( \phi_i(t) \), their derivatives and estimate non-linearities.

5. APPLICATIONS

5.1 System with viscous damping and cubic stiffness non-linearities

The first example considered involves a single degree of freedom system with viscous damping and cubic stiffness non-linearities. The equation of motion is of the kind:

\[ \ddot{x}(t) + \mu \dot{x}(t) + \omega_n^2 x(t) + \eta_3 x^3(t) = 0 \]  

(26)
where $\mu = 0.7 \, s^{-1}$; $\omega_n = 20 \, rad.s^{-1}$; $\eta_3 = 500 \, (m.s)^{-2}$. The system was simulated using a fourth order Runge-Kutta procedure with initial displacement $x(0) = 0.5 \, m$. The number of data samples was 2048 and the time record 12 s. The signal was additionally corrupted by zero mean Gaussian noise with SNR = 15 dB. Table 1 shows the results of the identification obtained using the direct parameter estimation method. In this table, we have the parameters identified by a least squares procedure and a total least squares [10] method. The errors between exact values and identified values are also given. The identification is satisfactory.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Exact values</th>
<th>Identified values by Least Squares</th>
<th>Errors %</th>
<th>Identified values by Total Least Squares</th>
<th>Errors %</th>
</tr>
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<tr>
<td>$\mu_1 , (s^{-1})$</td>
<td>0.7</td>
<td>0.72</td>
<td>3.2</td>
<td>0.72</td>
<td>3.2</td>
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<td>$\omega_n , (rad.s^{-1})$</td>
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<td>20.01</td>
<td>$4 \times 10^{-2}$</td>
<td>20.01</td>
<td>$4 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\eta_3 , (m.s)^{-2}$</td>
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<td>495.77</td>
<td>0.8</td>
<td>495.77</td>
<td>0.8</td>
</tr>
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</table>

Table 1. Estimation of parameters using the direct parameter estimation method for a system with viscous damping and cubic stiffness non-linearities

Figure 1 (a) shows the response of the noisy signal. To improve the resolution of the wavelet transform we choose the modified Morlet wavelet function and determine the parameter $N$ by minimization of the wavelet entropy. As shown in Figure 1(b) the minimal value of $N$ is 13. This value is the optimal value used in the modified Morlet wavelet. Figure 1(c) shows the amplitude of the wavelet transform for the noisy signal and its ridge when $N=13$.

Figure 1. Time response and amplitude of the wavelet transform for the system with viscous damping and cubic stiffness non-linearities

From the ridge we extract the skeleton of the wavelet transform. The real part of the skeleton gives the recovered signal. Figure 2(a) shows the comparison between the theoretical free response obtained from (26) in solid line and the identified free response obtained from the skeleton of the wavelet transform in dashed line. These plots are in good agreement. The envelope of the identified signal obtained from (23) is also plotted. However, note small disturbances at the beginning and end of the signal due to edge effects. Figure 2(b) shows the variations of $\dot{A}$ as a function of $A$. The characteristics of the analyzed system obtained from the skeleton of the wavelet transform given by a solid line, show very good agreement with the identified characteristics given by a dashed line and obtained from the oscillator model. This figure clearly shows the presence of viscous damping in the system since we obtain a straight line. From the slope of this line we estimate the damping coefficient $\mu_1$ since $\dot{A} = c_1 A = -\mu_1 A / 2$. Figure 2(c) shows the backbone curve of the analyzed system (solid line) obtained from the skeleton of the wavelet transform and the backbone curve of the identified system (dashed line). From this plot we obtain the frequency $f_n$ for $A=0$.

Figure 2. Theoretical and recovered time response; $\dot{A} = f(A)$ and backbone curves
Table 2 shows the RMSE between measured values of the instantaneous angular frequency $\dot{\phi}(t)$ obtained from the skeleton of the wavelet transform and identified values $\dot{\phi}(t_i)$ of the estimated oscillator model for different orders $q$ and values of $r$. It is clear that the optimal value is $q = 3$, we have then a cubic stiffness non-linearity. From (23) and (24) we obtain $\dot{\phi}(t) = \omega_n + 3\eta_3 t^2(t)/8\omega_n$ and we can compute from this expression the value of $\eta_3$. Table 2 shows the system parameter identification and the errors in percent between exact values and identified values for different levels of Gaussian noise added to the data. Even with important noise satisfactory results are obtained.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Exact</th>
<th>Identified, SNR=$\infty$</th>
<th>Identified, SNR=20 dB</th>
<th>Identified, SNR=15 dB</th>
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<td>$\mu_1$ (s$^{-1}$)</td>
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<td>0,702</td>
<td>0,702</td>
<td>0,690</td>
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<tr>
<td>$\omega_n$ (rad.s$^{-1}$)</td>
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<td>20,005</td>
<td>20,007</td>
<td>20,018</td>
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<tr>
<td>$\eta_3$ (m.s)$^{-2}$</td>
<td>500</td>
<td>502,044</td>
<td>497,265</td>
<td>495,175</td>
</tr>
</tbody>
</table>

Table 2. Parameter identification for the oscillator with viscous damping and cubic stiffness non-linearities

5.2 System with composite damping and non-linear stiffness

The second numerical example considered involves a single degree of freedom system with composite damping including linear and quadratic damping and a cubic stiffness non-linearity. The equation of motion is

$$\ddot{x} + \mu_1 \dot{x} + \mu_2 \dot{x} |x| + \omega_n^2 x + \eta_3 x^3 = 0$$

(27)

with $\mu_1=0,45$ s$^{-1}$; $\mu_2=0,12$ m$^{-1}$; $\omega_n=20$ rad.s$^{-1}$; $\eta_3=500$ (m.s)$^{-2}$. The free response of the system was simulated using a fourth order Runge-Kutta procedure with initial displacement $x(0)= 0,5$ m. The number of data samples was 2048 and the time record 9 s. Table 3 shows the results of the identification obtained using the direct parameter estimation method.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Exact values</th>
<th>Identified values by Least Squares</th>
<th>Errors %</th>
<th>Identified values by Total Least Squares</th>
<th>Errors %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$ (s$^{-1}$)</td>
<td>0,45</td>
<td>0,5</td>
<td>10,7</td>
<td>0,5</td>
<td>10,7</td>
</tr>
<tr>
<td>$\mu_2$ (s$^{-2}$)</td>
<td>0,12</td>
<td>0,11</td>
<td>4,6</td>
<td>0,10</td>
<td>9,1</td>
</tr>
<tr>
<td>$\omega_n$ (rad.s$^{-1}$)</td>
<td>20</td>
<td>20,04</td>
<td>0,2</td>
<td>17,8</td>
<td>10,8</td>
</tr>
<tr>
<td>$\eta_3$ (m.s)$^{-2}$</td>
<td>500</td>
<td>483</td>
<td>3,4</td>
<td>1631</td>
<td></td>
</tr>
</tbody>
</table>

Table 3. Estimation of parameters using the direct parameter estimation method for the system with composite damping and non-linear stiffness

The results obtained by least squares are superior to these obtained by total least squares. Figure 3 (a) shows the amplitude of the wavelet transform and its ridge. From the ridge we extract the skeleton of the wavelet transform. Figure 3(b) gives a comparison between real part of the wavelet transform skeleton obtained from the ridges and the theoretical free response. The envelope of the identified signal is also plotted. This, apart the ends, shows perfect match of signals.
Figure 3. Amplitude of the wavelet transform; theoretical and recovered response method for the system with composite damping and non-linear stiffness.

Figure 4(a) shows the backbone curves and the variations of $\dot{A}$ as a function of $A$ for the measured and identified mechanical system. These results clearly display the non-linear characteristics of the system. The natural frequency is estimated from the backbone curve for $A=0$; the order $q$ and the coefficient of non-linear stiffness are obtained using the $RMSE(\phi)$ procedure described previously. Damping coefficients are estimated from a polynomial interpolation of the continuous parabola obtained in Figure 4 (b). The characteristics obtained from the skeleton of the wavelet transform given by a solid line, show very good agreement with the theoretical characteristics identified from the oscillator model and given by a dashed line.

The estimated values of parameters and their errors for different levels of noise are presented in Table 4.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Exact values</th>
<th>Identified values by wavelet transform</th>
<th>Errors %</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1(s^{-1})$</td>
<td>0,45</td>
<td>0,473</td>
<td>5,1</td>
</tr>
<tr>
<td>$\mu_2(s^{-2})$</td>
<td>0,12</td>
<td>0,118</td>
<td>1,7</td>
</tr>
<tr>
<td>$\omega_n(\text{rad.s}^{-1})$</td>
<td>20</td>
<td>20,01</td>
<td>0,05</td>
</tr>
<tr>
<td>$\eta_3(\text{m.s})^{-2}$</td>
<td>500</td>
<td>500,4</td>
<td>$7\times10^{-2}$</td>
</tr>
</tbody>
</table>

Table 3. Estimation of parameters using the wavelet transform method for the system with composite damping and non-linear stiffness.

These results are satisfactory and superior to the results obtained from the direct parameter estimation method.
5. CONCLUSION

A direct parameter estimation method and a wavelet transform method have been used for identification, classification and quantification of non-linearities in vibrating systems. When the number of parameters grow the direct parameter estimation method gives bad results. It is probably due to the numerical estimation of velocity and acceleration obtained by numerical derivation of the displacement. The errors on velocity grow with the power \( p \) of non-linearity on damping. The wavelet transform method gives better results. The procedure is based on the ridges and skeletons of the wavelet transform. While a great number of authors use the Morlet wavelet function as analyzing function, the modified Morlet wavelet function has been used in this paper. Its time and frequency resolution can be altered by adjusting the value of a parameter \( N \). The effectiveness of the proposed method has been demonstrated using numerical results for a SDOF. The proposed approach is particularly suitable for the identification and quantification of non-linearities of noisy vibrating systems. It was found that the identification was still good, even with a signal to noise ratio of 15 dB, although for better results the signal to noise ratio should be greater than 10 dB. The procedure presented was also tested experimentally on a clamped beam where non-linearities on damping and stiffness have been added. Results concerning this experimental test will be presented at the conference.

REFERENCES