Modeling guided wave propagation for structural health monitoring applications

Ivan Bartoli*, Alessandro Marzani**, Howard Matt*, Francesco Lanza di Scalea*, Erasmo Viola**

* Dept. of Structural Engineering, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0085
** Dipartimento di Ingegneria delle Strutture, dei Trasporti, delle Acque, del Rilevamento, del Territorio (DISTART), Universita’ degli Studi di Bologna, Via Risorgimento 2, 40136 Bologna, Italia

Abstract

In this paper a Semi-Analytical Finite Element (SAFE) method for modeling wave propagation in waveguides of arbitrary cross-section is discussed. The method requires the finite element discretization of the cross-section of the waveguide by assuming harmonic motion along the wave propagation direction. The general SAFE technique is extended to account for viscoelastic material damping by allowing for complex stiffness matrices for the material. Phase velocity, energy velocity, attenuation, and cross-sectional mode shapes are obtained by solving an eigenvalue problem. Knowledge of these properties is important in any structural health monitoring attempt that uses ultrasonic guided waves. The proposed SAFE formulation is applied to several examples, including isotropic viscoelastic plates, composite-to-composite adhesive joints and railroad tracks.

1. Introduction

Guided ultrasonic waves provide a highly efficient method for the non-destructive evaluation (NDE) and the structural health monitoring (SHM) of solids with finite dimensions. Compared to ultrasonic bulk waves, guided waves provide larger monitoring ranges and the complete coverage of the waveguide cross-section. Compared to global vibrations, guided waves provide increased sensitivity to smaller defects due to the larger frequencies. These advantages can be fully exploited only once the complexities of guided wave propagation are unveiled and managed for the given test structure. These complexities include the existence of multiple modes, the frequency-dependent velocities (dispersion), and the frequency-dependent attenuation. For example, the knowledge of the wave velocity is important for mode identification. Similarly, the knowledge of those mode-frequency combinations propagating with minimum attenuation losses helps maximizing the inspection coverage.

Semi-Analytical Finite Element (SAFE) methods have emerged for modeling the guided wave propagation numerically as an alternative to the "exact" methods based on the superposition of bulk waves – SPBW, that include the popular matrix-based methods [1]. Motivations for the numerical methods include the necessity for modeling a large number of layers such as composite laminates and that of modeling waveguides with arbitrary cross-section for which exact solutions do not generally exist. In addition, when complex wavenumbers are part of the solution such as in the case of leaky and/or damped waveguides, the exact SPBW methods require iterative bi-dimensional root searching algorithms that may miss some of the solutions [1].

The general SAFE approach for extracting dispersive solutions uses a finite element discretization of the cross-section of the waveguide alone. The displacements along the wave propagation direction are conveniently described in an analytical fashion as harmonic exponential functions. Thus only a bi-dimensional discretization of the cross-section is needed, with considerable computational savings compared to a three-dimensional discretization of the entire waveguide. The SAFE solutions are obtained in a stable manner from an eigenvalue problem, and thus do not require the root searching algorithms used in SPBW approaches.

A SAFE method for waveguides of arbitrary cross-section was demonstrated for the first time in 1973 [2, 3]. In these works dispersive solutions were obtained for the propagative modes only (i.e. real wavenumbers only). The same technique was used a decade later [4] to calculate both propagative modes and nonpropagative,
evanescent modes (complex wavenumbers) for anisotropic cylinders. While the evanescent modes do not transport any energy along the structure, they are important from a theoretical viewpoint to satisfy the boundary conditions. More recently, SAFE methods confined to obtaining the propagative solutions were applied to wedges [5], rods and rails [6, 7]. An approximation of the method in [6, 7] was also implemented in a standard finite element package by imposing a cyclic axial symmetry condition [8]. Laminated composite waveguides were studied by SAFE methods in [9, 10].

The focus of previous SAFE works was obtaining propagative and evanescent modes in undamped waveguides. A need exists to extend this technique to account for material damping. One very recent work [11] demonstrates a SAFE application to damped, viscoelastic composite laminates. In this reference a damping loss factor was estimated indirectly from the power dissipated by the wave. However, the formulation in [11] still does not allow for the calculation of the true wave attenuation since the governing stiffness matrix was assumed real. The present study extends the SAFE method for modeling dispersive solutions in waveguides of arbitrary cross-sections by accounting for material damping. This extension is particularly relevant for NDE/SHM applications on high-loss materials such as viscoelastic fiber-reinforced polymer composites. When accounting for damping, the energy velocity, rather than the conventional group velocity, is calculated along with the frequency-dependent attenuation of the modes. Various examples are shown, including an isotropic plate, a composite-to composite adhesive joint and a railroad track.

2. Viscoelastic models for wave propagation
A linear viscoelastic model was used in the SAFE formulation proposed in order to model the material damping. Linear viscoelasticity can be modeled by allowing complex components in the material's stiffness matrix as follows

\[ C^* = C' - iC'' \]  \hspace{1cm} (1)

In Eq. (1) \( C' \) contains the storage moduli and \( C'' \) contains the loss moduli.

In practice, the matrix \( C^* \) can be expressed as a combination of the elastic stiffness tensor, \( C' \), and the viscosity tensor \( \eta \). The coefficients of the viscosity tensor are typically measured at a single frequency value that is the characterization frequency.

The hysteretic model, well-established in ultrasonic NDE, was considered in this study to represent material damping. The imaginary component of the stiffness matrix in Eq. (1) is frequency independent, thus:

\[ C^* = C' - i\eta \]  \hspace{1cm} (2)

As a consequence, the hysteretic stiffness matrix has to be determined only once for the entire frequency range examined.

The wave attenuation, defined as the loss per unit distance travelled, is commonly modelled as proportional to the frequency times the imaginary part of the stiffness matrix \( C'' \) [18]. Therefore the attenuation is a linear function of the frequency in the case of the hysteretic model.

3. SAFE mathematical framework
The mathematical model is presented here for the case of a waveguide immersed in vacuum, as shown in Fig. 1(a). This figure refers to an infinitely long waveguide with arbitrary cross-section. The wave propagates along direction \( x \) with wavenumber \( \xi \) and frequency \( \omega \). The cross-section lies in the \( y \)-\( z \) plane. In Fig. 1(b) the waveguide is an infinite plate that can generally be composed of anisotropic viscoelastic materials as well as orthotropic layers. The displacement, stress and strain field components at each point of the waveguide are expressed by:

\[ u = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}, \quad \sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z \\ \varepsilon_{xy} & \varepsilon_{yx} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{zx} & \varepsilon_{zz} \end{bmatrix} \]  \hspace{1cm} (3)

Equations of motion for the cross section are obtained by the Hamilton’s principle

\[ \delta H = \int_{t_i}^{t_f} \delta (\phi - K) \, dt = 0 \]  \hspace{1cm} (4)

In Eq. (4) \( \phi \) is the strain energy and \( K \) is the kinetic energy. The strain energy is given by:

\[ \phi = \frac{1}{2} \int_V \varepsilon^T C^* \varepsilon \, dV \]  \hspace{1cm} (5)

where the upper script T means a transpose vector and \( V \) is the volume. The kinetic energy is given by:

\[ K = \frac{1}{2} \int_V \rho \dot{u}^T \dot{u} \, dV \]  \hspace{1cm} (6)
where \( \rho \) is the mass density and the dot represents a time derivative.
In general, the nonconservative form of Hamilton’s principle should be used to account for dissipation. However, the following analysis adopts a simplified approach that assumes a conservative waveguide; the resulting imaginary cross-sectional strain energy distribution is used to estimate the power dissipated by the section via imaginary wavenumbers. The assumption is valid if the cross-sectional strain energy distribution of a propagating wave is not significantly modified by increasing levels of damping [11].

The harmonic motion is represented by introducing a displacement field that is harmonic along the propagation direction, \( x \) while shape functions are used to interpolate the displacement inside the element domain \( \Omega_e \) in the cross-sectional plane \( y-z \). The final expression of the displacement field is:

\[
\begin{bmatrix}
  u_x(x,y,z,t) \\
  u_y(x,y,z,t) \\
  u_z(x,y,z,t)
\end{bmatrix} = \begin{bmatrix}
  \sum_{k=1}^{n} N_k(y,z) U_{x_k} \\
  \sum_{k=1}^{n} N_k(y,z) U_{y_k} \\
  \sum_{k=1}^{n} N_k(y,z) U_{z_k}
\end{bmatrix} e^{i(\xi x - \omega t)} = N(y,z) q(e) e^{i(\xi x - \omega t)}
\] (7)

where \( i = \sqrt{-1} \) is the imaginary unit, \( N_k(y,z) \) are the shape functions, and \( U_{x_k}, U_{y_k}, U_{z_k} \) represent the nodal unknown displacements in the \( x, y \) and \( z \) directions. Shape functions and nodal unknowns can be summarized in the correspondent vectors \( N(y,z) \) and \( q(e) \).

The displacement expression of Eq.(7) are used to compute the strain vector \( \boldsymbol{\varepsilon} \) by the standard strain displacement relations [15]. Stress components are obtained by the constitutive laws \( \sigma = C^{*} \varepsilon \). Stress, strain and displacement components are finally expressed in terms of the nodal unknowns vector \( q(e) \). By substituting them in Eq. (4), performing some manipulations and using standard finite element assembling procedure the following eigenvalue problem is obtained [7]:

\[
[\mathbf{A} - \xi^{2} \mathbf{B}]_{2M} \mathbf{Q} = \mathbf{0}
\] (8)

where \( \mathbf{A} \) and \( \mathbf{B} \) are complex matrices depending on the temporal frequency \( \omega \) and \( \mathbf{Q} \) is the eigenvector corresponding to the eigenvalue \( \xi \). From Eq. (8), at each frequency \( \omega \), \( 2M \) eigenvalues \( \xi_m \) and consequently \( 2M \) eigenvectors are obtained. The eigenvectors are the \( M \) forward and the corresponding \( M \) backward modes. The eigenvalues are complex numbers. The phase velocity can be evaluated by \( c_{ph} = \omega / \xi_{real} \) and the attenuation, in Nepers per meter, by \( \xi_{lim} \), where \( \xi_{real} \) and \( \xi_{lim} \) are the real and imaginary parts of the wave number.

As reported in [14], the group velocity definition is not valid in damped waveguides. In this case the wavenumber is complex and the differentiation \( \frac{\partial}{\partial \xi} \) is no longer possible. If the differentiation is made with respect to the real part of the complex wavenumber, then the group velocity calculation yields non-physical solutions such as infinite velocities at some locations of the dispersion curves. The energy velocity, \( V_e \), is the appropriate property for damped media. The definition of the energy velocity can be found in classical textbooks [15]. The expression used in the present work is:
\[ V_\omega = \frac{1}{\Omega} \int \mathbf{P} \cdot \mathbf{x} d\Omega - \frac{1}{T} \int \left( \frac{1}{\Omega} e_{\text{tot}} d\Omega \right) dt \]

where \( \mathbf{x} \) is the unit vector along the wave propagation direction \( \mathbf{x} \), \( 1/T \int (...) dt \) denotes the time average over one period \( T \), \( e_{\text{tot}} \) is the total energy density (kinetic and potential), and \( \mathbf{P} \) represents the time averaged Poynting vector (real part only). The time averaged Poynting vector can be calculated as:

\[ \mathbf{P} = \frac{1}{2} \operatorname{Re} (\sigma \mathbf{u}^*) \]

where \( \sigma \) is the classical 3x3 stress tensor, and \( \mathbf{u}^* \) is the complex conjugate of the particle velocity vector. The numerator in Eq. (9) is the average power flow carried by a mode in the wave propagation direction over a unit period of time.

The denominator in Eq. (9) can be evaluated by introducing the expressions of the time averaged energy for the kinetic component, \( \langle e_k \rangle_T \), and the potential component, \( \langle e_p \rangle_T \), following the formulation in [12]:

\[ \langle e_k \rangle_T = \frac{\omega^2}{4} \rho \mathbf{u}^* \mathbf{u} , \quad \langle e_p \rangle_T = \frac{1}{4} \mathbf{C}^t \mathbf{\varepsilon} \]

where the constants \( \frac{1}{4} \) result from the time integration over the period \( T \). Eqs. (11) can be evaluated and substituted in Eq. (9) once the element nodal displacements are calculated from the eigenvalue problem and the displacement and strain fields are reconstructed.

4. Results

4.1. Plate systems

The general plate system (Fig. 1(b)) consists of an arbitrary number \( n \) of orthotropic layers stacked along the \( z \) direction. The origin of the reference Cartesian system \( (x, y, z) \) is located at the top of the layered plate and each layer lies parallel to the \( x-y \) plane. The plates considered in this study have an infinite length in the width direction, \( y \). Thus the two-dimensional cross-section that needs to be interpolated by finite elements reduces to a single line through the plate thickness. Mono-dimensional quadratic elements were used for the line discretization (Fig. 1(c)). In general, each element can have three degrees of freedom (d.o.f.) per node, associated to the displacements \( u_x \), \( u_y \) and \( u_z \). For isotropic plates, the Lamb modes polarized in the \( x-z \) plane are de-coupled from the shear horizontal (SH) modes that are, instead, polarized in the \( x-y \) plane. The de-coupling holds for orthotropic plates when the wave propagation direction is along a direction of principal material symmetry. Consequently, in these cases the number of d.o.f. of each analysis can be reduced by solving for the Lamb modes and for the SH modes separately (thus considering only \( u_x \) and \( u_z \) for the former modes, and only \( u_y \) for the latter modes). For orthotropic plates with an arbitrary wave propagation direction or for laminated composite plates, the Lamb and the SH modes are coupled and thus these solutions must be found simultaneously.

4.1.1. Viscoelastic isotropic plate

The first system examined is a viscoelastic isotropic High Performance PolyEthylene (HPPE) plate in vacuum. This plastic material has a relatively high damping. This example was chosen because fully studied in [14] by using the software DISPERSE that is based on a SPBW method. The physical and geometric characteristic of the HPPE plate are the same as those in [14]: density \( \rho = 953 \text{ kg/m}^3 \), thickness \( h = 12.7 \text{ mm} \), longitudinal bulk velocity \( c_L = 2344 \text{ m/s} \), shear bulk velocity \( c_T = 953 \text{ m/s} \), longitudinal bulk wave attenuation \( \kappa_L = 0.055 \text{ Np/wavelength} \) and shear bulk wave attenuation \( \kappa_T = 0.286 \text{ Np/wavelength} \).

For the SAFE modeling, the complex bulk velocities for the viscoelastic material must be first calculated as follows:

\[ \tilde{c}_{LT} = c_{LT} \left( 1 + i \frac{\kappa_{LT}}{2 \pi} \right) \]

The complex Young’s modulus, \( \tilde{E} \), and complex Poisson's ratio, \( \tilde{\nu} \), can be obtained as:

\[ \tilde{E} = \rho \tilde{c}_T \left( \frac{3 \tilde{c}_L^2 - 4 \tilde{c}_T^2}{\tilde{c}_L^2 - \tilde{c}_T^2} \right) , \quad \tilde{\nu} = \frac{1}{2} \left( \frac{\tilde{c}_L^2 - 2 \tilde{c}_T^2}{\tilde{c}_L^2 - \tilde{c}_T^2} \right) \]

The complex Lame’ constants

\[ \lambda^* = \frac{\tilde{E} \tilde{\nu}}{(1 + \tilde{\nu})(1 - 2 \tilde{\nu})} , \quad \mu^* = \frac{\tilde{E}}{2(1 + \tilde{\nu})} \]
are finally used to calculate the complex viscoelastic stiffness matrix [16]. Forty, quadratic mono-dimensional elements with three nodes per element were used for the SAFE discretization. For the Lamb wave solutions, only the $u_x$ and $u_y$ d.o.f. were used resulting in a total of 162 d.o.f. The resulting Lamb wave solutions are shown in Fig. 2(a) and 2(b). The energy velocity values, Fig. 2(a), were obtained from Eq. (9). The attenuation values are shown up to 500 Np/m in Fig. 2(b). The frequency range presented is coincident with the one considered in [14]. In these reference, however, some of the solutions of the attenuation curves are missing resulting in interrupted or discontinuous branches. This is a consequence of the difficulty of the searching algorithm based on the SPBW method to converge. The SAFE results in Fig. 2 show no missing roots. Compared to an undamped elastic plate where no solutions exist below the cut-off frequencies, all modes in Fig. 2 have solutions that extend to the origin of the frequency axis. This is the result of the real wavenumber that is now associated to the formerly nonpropagative roots of the undamped case. Below the undamped cut-off frequencies, the damped solutions are characterized by large attenuation values and small energy velocity values. Although these portions have an interesting theoretical significance, they have little practical use in NDE/SHM. If needed, the “nonpropagative” branches can be easily deleted from the dispersion curves by thresholding either the attenuation or the energy velocity values. Highlighted in Fig. 2 is the symmetric mode, $m$, which has the lowest attenuation above 165 kHz. Because of the low attenuation, this mode was examined in detail in [14]. As confirmed in this reference, both phase and energy velocities for $m$ tend to the bulk longitudinal velocity as the frequency increases, since the dominant displacements are along the wave propagation direction.

4.1.2. Composite-to-composite adhesively-bonded joint

This section presents SAFE results relative to ongoing efforts at UCSD aimed at the development of an on-board SHM system for Unmanned Aerial Vehicles (UAVs) based on integrated sensors and ultrasonic guided waves [17]. The monitoring is being targeted to the adhesive bond between the UAV composite wing skin and the tubular composite spar shown in the drawing of Fig. 3(c). The spar runs along the lengthwise direction of the wing. The wing skin under investigation is a T300/5208 carbon-epoxy laminate with a stacking sequence [0/±45/0]s and a thickness of 0.133mm per lamina. The 0-deg direction is parallel to the spar lengthwise direction. The wing skin was modeled by the usual rotation matrices [15, 16]. The spar is a cross-ply tubular section made of T800/924 and having a total wall thickness of 5.235 mm. In the model, the spar was considered as one equivalent viscoelastic orthotropic layer. The adhesive layer had a typical thickness of 0.203 mm. One quadratic element was used for each lamina of the skin and for the bond layer, whereas five elements were used for the spar wall. The hysteretic viscoelastic model was used for each of the components.

The on-board sensor disposition is such that the wave is generated and detected on the wing skin on either side of the joint. The wave propagation direction is perpendicular to the spar, along direction x in the drawing of Fig. 3. Any degradation in the bond condition can then be monitored by measuring changes in the strength of the ultrasonic transmission through the joint.

SAFE dispersion results are presented in Fig. 3 for the disbonded skin-to-spar interface, the most extreme bond degradation that was considered in this study. The specific material properties assumed for the various layers of the joint are summarized in Table 1 and 2. The solutions in Fig. 3 show modes whose energy is mainly concentrated within the wing skin above the bondline (identified in the figure by $S_{0,\text{plate}}, A_{0,\text{plate}}, SH_{0,\text{plate}}$, etc.), and modes whose energy is mainly concentrated within the spar below the bondline (identified by $S_{0,\text{spar}}, A_{0,\text{spar}}$.

**Figure 2.** Dispersion results (Lamb modes) for a 12.7mm thick, viscoelastic HPPE plate in vacuum: (a) energy velocity, (b) attenuation below 500 Np/m. Low-attenuation symmetric mode $m$. 
SH0,spar, etc.). The former modes closely match those that would be supported by the wing skin alone (identified by s0, a0, sh0 and represented by open dots in Fig. 3). The match between the “skin” modes of the disbonded joint and the pure single-skin modes becomes closer as the frequency increases (compare, for example, A0,plate and a0). One implication is that waves generated on the wing skin outside of the joint will be transferred very efficiently across the disbonded interface through one of the “skin” modes. Consequently, the occurrence of a disbond can be detected by an increased strength of ultrasonic transmission compared to a regularly-bonded joint.

Table 1. Elastic and viscous properties for the UAV wing skin-to-spar joint.a

<table>
<thead>
<tr>
<th>Layer</th>
<th>C''11</th>
<th>C''12</th>
<th>C''13</th>
<th>C''22</th>
<th>C''23</th>
<th>C''33</th>
<th>C''44</th>
<th>C''55</th>
<th>C''66</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wing skin lamina</td>
<td>135</td>
<td>5.70</td>
<td>5.70</td>
<td>14.2</td>
<td>8.51</td>
<td>14.2</td>
<td>2.87</td>
<td>4.55</td>
<td>4.55</td>
</tr>
<tr>
<td>Spar wall</td>
<td>88.0</td>
<td>5.45</td>
<td>5.09</td>
<td>8.0</td>
<td>5.09</td>
<td>11.3</td>
<td>4.64</td>
<td>4.64</td>
<td>6.00</td>
</tr>
<tr>
<td>Disbond</td>
<td>0.070</td>
<td>0.069</td>
<td>0.069</td>
<td>0.070</td>
<td>0.069</td>
<td>0.070</td>
<td>0.00012</td>
<td>0.00012</td>
<td>0.00012</td>
</tr>
</tbody>
</table>

aElastic constants in GPa. *from Ref. [18].

Table 2. Bulk ultrasonic velocities and attenuations of the UAV wing skin-to-spar interface layer.

<table>
<thead>
<tr>
<th>Layer</th>
<th>CL [m/s]</th>
<th>CT [m/s]</th>
<th>KL [Np/λ]</th>
<th>KT [Np/λ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular bond</td>
<td>2410</td>
<td>1210</td>
<td>0.149</td>
<td>0.276</td>
</tr>
<tr>
<td>Disbond</td>
<td>241</td>
<td>12.1</td>
<td>1.497</td>
<td>2.763</td>
</tr>
</tbody>
</table>

Figure 3. Dispersion results for UAV wing skin-to-spar adhesive joint (c) with a disbonded interface for waves propagating perpendicularly to the spar lengthwise direction: (a) energy velocity, (b) attenuation. ooo single skin modes.
4.2. Arbi trary cross-sections: viscoelastic railroad track
The purpose of this section is to demonstrate the applicability of the SAFE approach to waveguides of arbitrary cross-sections that cannot be solved by exact methods. The case treated is that of a railroad track. Knowledge of the dispersive behavior of guided waves in rails is relevant for the purpose of train noise reduction at low frequencies, below 6 kHz [19], and for long-range NDE defect detection at high frequencies, up to 50 kHz [20].

The rail considered is a typical 115-lb A.R.E.M.A. section, modeled as an isotropic material with hysteretic damping, and having the following properties:
\[
\rho = 7932 \text{ kg/m}^3, \quad c_L = 5960 \text{ m/s}, \quad c_T = 3260 \text{ m/s}, \quad \kappa_L = 0.003 \text{ Np/wavelength} \quad \text{and} \quad \kappa_T = 0.043 \text{ Np/wavelength.}
\]
The rail cross-section has a complex geometry with one vertical axis of symmetry. The mesh, shown in Fig. 4 and generated by Matlab’s “pdetool,” used 81 nodes for 106 triangular elements with linear interpolation displacement functions.

The dispersion results are shown in Fig. 4 up to a frequency of 50 kHz. The complexity of the modes is evident in these plots. Notice that no prior solutions for the attenuation values are available from the literature in this frequency range, since previous wave propagation models of rails did not include damping, with the exception of low-frequency (< 6 kHz) studies [19]. Knowledge of the high-frequency attenuation, however, is important to identify low-loss mode-frequency combinations that can provide truly long-range defect detection.

5. Discussion and conclusions
In this paper a general SAFE method was proposed and derived for modeling wave propagation in waveguides of generally arbitrary cross-sections. The main innovation over SAFE models proposed in the past is accounting for viscoelastic material damping, and thus representing the energy velocity and the attenuation curves. The hysteretic viscoelastic model (frequency independent) was used in the formulation. The energy velocity values are obtained at each individual wavenumber-frequency solution, without the need for tracking the modes and without considering adjacent solutions in finite difference calculations. The knowledge of the dispersive wave properties is relevant for NDE/SHM testing for identifying propagating modes, locating defects, as well as exciting low-loss mode-frequency combinations for increased ranges.
The method was validated on various examples, some of which were examined previously by SPBW methods. The examples included viscoelastic isotropic plates, composite-to-composite adhesive bonds in UAV wings, and railroad tracks.

The approximation of any SAFE method depends on the discretization of the waveguide’s cross-section. In the absence of rigorous convergence criteria, the rule of thumb should be to have the necessary number of elements to properly represent the cross-sectional mode shapes and stress/strain distributions of the problem for the highest frequency of interest.

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References