Reduced models of thermo-mechanical systems for efficient analysis in the Concurrent Design Facility at the European Space Agency

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Abstract
In order to include realistic models early in the design of space applications and in the multi-disciplinary review iterations of the concepts, reduction strategies have been used to build libraries of models that can be rapidly accessed by experts in a design team. In the present paper we explain how, in thermo-mechanical problems, the reduction space considered to construct the reduced model can been enriched and improved by considering the quasi-static structural response associated to the thermal modes.¹

1. Introduction
At the Technical Centre of the European Space Agency, the Concurrent Design Facility (CDF) is where a multidisciplinary team of experts work together in a concurrent way and in real time to realize the initial conception of satellites, programs and space instruments and to review industrial concepts. The trend in the CDF is also to support the later phases of the space projects where the analysis and simulation models need to be finer. One important aspect of such later phase activity is the structure of the system, including not only its static and dynamic mechanics, but also its interaction with other non-mechanical fields such as thermal effects and optical performance for instance.
In order to allow efficient, concurrent and close to real time interactions between experts and their models it is important to have access to rapid and accurate analysis in order to perform design iterations to reach an optimal layout.
In the framework of the project CoDe:Oofelie::CD (Oofelie for Concurrent Design using Integrated CAE and Reduction Models) new functionalities were added to the MS-Excel® environment used for the integration and exchange of data at the CDF. The new tool allows the designer to access pre-computed and parameterized reduced models in order to perform rapid multidisciplinary analyses at a relatively high level of accuracy. The design models and submodels are obtained by applying reduction techniques similar to the Craig-Bampton [1] method to the detailed multiphysical models available from detailed designs or from industrial models.
The project has investigated how the reduction strategy can be integrated in the design environment. The effectiveness of the approach has been illustrated both on academic and industrial examples.

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In this project the work on the software consisted in making the reduced models available through the designer interface. From a numerical point of view one research issue consisted in constructing reduced model for parameterized subcomponents of the design. Another challenge consisted in building accurate low order reduced models that can properly handle the multiphysical character of the problems handled in the design of space applications. This is the topic outlined in this paper.

In this paper we will discuss an efficient way to build a reduction basis for thermo-mechanical systems where the thermal time-constant is significantly larger than the time constant of the structure. First we shortly summarize the concepts of model reduction and component mode synthesis. Then we outline the thermo-mechanical equations and discuss a special enrichment of the internal modes in order to efficiently represent the dynamic response of a structure to a thermal load. Finally we illustrate the efficiency of the proposed basis on a cantilever beam model.

2. Concept of model reduction

Let us consider a finite element model characterized by the second order differential matrix equation in which \( q \) represents the vector of degrees of freedom (DOF), \( K \), \( D \) and \( M \) respectively the stiffness, damping and inertia matrices, and \( f \) the vector of forces

\[
M \ddot{q} + D \dot{q} + K q = f
\]  
Equation 1

The driving idea of reduction methods is to make an approximation on the model DOF and express them in terms of a linear combination of representative modes. It is clear that in order to obtain a reduced model, the number of modes must be much smaller than the number of DOF in the model.

The previous system is then rewritten in terms of “generalized” DOF \( \hat{q} \) that physically correspond to the amplitude of the considered modes. Mathematically this consists in building an approximation of the DOF vector according to a “trial” subspace defined by the modes, assumed to be linearly independent.

\[
q = R \hat{q}
\]  
Equation 2

In this approximation, \( R \) is a matrix which columns contain the vectors spanning the approximation space. Replacing this expression in the full set of equations lead to a the following equation

\[
MR \ddot{\hat{q}} + DR \dot{\hat{q}} + KR \hat{q} = f + r
\]  
Equation 3

The residual load \( r \) appears because, due to the approximation on DOF, the equation can not be satisfied exactly in general (unless the approximation space represented by \( R \) spans the exact solution, which is never the case in practice). It can be noted that once an approximate solution is computed (as described below) the equation above can always be used to evaluate the associated error in the exact problem.

Clearly, there are more equations to satisfy than degrees of freedom in the reduced space. Obviously the idea is to find the set of reduced DOF that minimizes the residue \( r \).

In the Petrov-Galerkin method one wants to ensure that the residue is null when projected in a given subspace of the problem. Hence in order to get an approximation of the problem, the equation is also projected onto a “test” subspace:

\[
\bar{R}^T \{ MR \ddot{\hat{q}} + DR \dot{\hat{q}} + KR \hat{q} \} = \bar{R}^T \{ f + r \}
\]  
Equation 4

and setting the projection of the residue to zero, yielding
\[
\ddot{M}\ddot{q} + \dot{D}\dot{q} + \ddot{K}q = \ddot{f}
\]

Equation 5

with the reduced terms

\[
\dot{M} = \tilde{R}^T M R \\
\dot{D} = \tilde{R}^T D R \\
\ddot{K} = \tilde{R}^T K R \\
\ddot{f} = \dot{R}^T f
\]

Equation 6

It is clear that the choice of representative trial and test bases, \( R \) and \( \tilde{R} \), is the key of the method and is directly related to the quality of the approximation. Indeed, if the trial base defines a subspace that contains the exact solution of the problem, then the residual term is null and the reduced model is exact. On the other hand one needs to use a test basis \( \tilde{R} \), such that the error remaining after solving the reduced problem (the error being in the space orthogonal to \( \tilde{R} \)) does not significantly alter the solution.

Often (such as in symmetric problems) the test and trial bases are taken as identical. In that case the reduction method is in fact a simple Galerkin technique (which is equivalent to a Rayleigh-Ritz method for dynamic problems). In that case the reduction technique can also be interpreted as a virtual work principle (see for instance [2]).

3. Dynamic condensation and component mode synthesis

The well known Guyan-Irons static condensation algorithm [3], can be seen as a projection method using static deformation shape as trial and test base. Nevertheless, in the case of dynamic analysis, the dynamic behaviour of the internal domain of the model, that means the condensed DOF, is not taken into account.

The improvement proposed by Craig & Bampton [1, 4] consists in adding to the trial and test bases a given number of modes directly related to the dynamic behaviour of the internal part of the model. Craig proposed to consider the internal eigenmodes \( \Phi \) of the model, computed by constraining the retained DOF to zero. These modes indeed represent the dynamic neglected in the quasi-static modes.

In a general way, the modes added to the trial and test bases, respectively \( \Phi \) and \( \tilde{\Phi} \), can be different. Nevertheless, when using internal eigen modes, the common practice consists in considering the same modes in the two bases.

In the Craig-Bampton method the following trial and test bases are considered\(^2\)

\[
q = \begin{pmatrix} q_r \\ q_c \end{pmatrix} = R \begin{pmatrix} q_r \\ \gamma \end{pmatrix} = R\ddot{q}
\]

\[
R = \begin{bmatrix} I \\ -K^{-1}K_c \end{bmatrix} \\
\tilde{R}^T = \begin{bmatrix} I - K_rK_c^{-1} \\ 0 \end{bmatrix} \
\]

Equation 7

where we denote with a subscript \( c \) the degrees of freedom that are condensed (i.e. the ones internal to a substructure) and by \( r \) the remaining ones (the degrees of freedom on an interface). The equation of the reduced model then becomes

\(^2\) Note that in the original Craig-Bampton the test and trial bases are identical. Here we generalize the method by allowing different bases.
\[
\ddot{\mathbf{q}} + \dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}
\]

Equation 8

\[
\mathbf{K} = \begin{bmatrix}
K_{rr} - K_{rc}K_{cc}^{-1}K_{cr} & 0 \\
0 & \tilde{\Phi}^\top K_{cc}\Phi
\end{bmatrix}
\]

\[
\mathbf{D} = \begin{bmatrix}
D_{rr} + K_{rc}K_{cc}^{-1}D_{cc}K_{cc}^{-1}K_{cr} & \left(D_{rc} - K_{rc}K_{cc}^{-1}D_{cc}\right)\Phi \\
\tilde{\Phi}^\top\left(D_{cr} - D_{cc}K_{cc}^{-1}K_{cr}\right) & \tilde{\Phi}^\top D_{cc}\Phi
\end{bmatrix}
\]

\[
\mathbf{M} = \begin{bmatrix}
M_{rr} + K_{rc}K_{cc}^{-1}M_{cc}K_{cc}^{-1}K_{cr} & M_{rc} - K_{rc}K_{cc}^{-1}M_{cc} \\
-K_{rc}K_{cc}^{-1}M_{cr} - K_{rc}K_{cc}^{-1}M_{cr} & \tilde{\Phi}^\top (M_{cr} - M_{cc}K_{cc}^{-1}K_{cr})
\end{bmatrix}
\]

Equation 9

4. Thermo mechanical models

In the case of thermo mechanical problems, the DOF representing the model contains structural and thermal DOF, respectively noted \(u\) and \(\theta\), and reordering the equations according to the physical nature of the DOF leads to the classical formulation of thermo-mechanical problems.

\[
\begin{bmatrix}
M_{uu} & 0 \\
0 & 0
\end{bmatrix}\ddot{\mathbf{u}}
+ \begin{bmatrix}
D_{uu} & 0 \\
D_{du} & D_{\theta\theta}
\end{bmatrix}\dot{\mathbf{u}}
+ \begin{bmatrix}
K_{uu} & K_{u\theta} \\
0 & K_{\theta\theta}
\end{bmatrix}\mathbf{u} = \begin{bmatrix}
f_u \\
f_\theta
\end{bmatrix}
\]

Equation 10

In the previous expression, \(M_{uu}\), \(D_{uu}\) and \(K_{uu}\) correspond to the structural inertia, damping and stiffness matrices, that are symmetric. The \(D_{\theta\theta}\) and \(K_{\theta\theta}\) matrices, that are also symmetric, can be seen as damping and stiffness terms, but describe respectively the heat capacitance and conduction effects.

The two remaining \(D_{du}\) and \(K_{u\theta}\) matrices characterize the thermo mechanical coupling. Due to these terms, the complete system is unsymmetric.

Partitioning again the DOF vector in terms of the retained and condensed parts, each of them containing both structural and thermal DOF, one can write

\[
\begin{bmatrix}
\mathbf{u}_r \\
\theta_r
\end{bmatrix}
= \begin{bmatrix}
\mathbf{u}_c \\
\theta_c
\end{bmatrix}
\]

Equation 11

The equations of the complete system can thus be reordered and rewritten in the following block-matrices form

\[
\begin{bmatrix}
M_{rr} & M_{rc} \\
M_{cr} & M_{cc}
\end{bmatrix}\ddot{\mathbf{q}}_r
+ \begin{bmatrix}
D_{rr} & D_{rc} \\
D_{cr} & D_{cc}
\end{bmatrix}\dot{\mathbf{q}}_r
+ \begin{bmatrix}
K_{rr} & K_{rc} \\
K_{cr} & K_{cc}
\end{bmatrix}\mathbf{q}_r
= \begin{bmatrix}
f_r \\
f_c
\end{bmatrix}
\]

Equation 12

with the sub-matrices.
5. Static and internal modes for thermo-mechanical models

We want to stress that, unlike for purely structural models, the static modes in the trial and test matrices are not identical when handling thermo-mechanical models. Indeed, in equation 26 the block $\begin{bmatrix} K_{ee} & K_{e\theta} \\ 0 & K_{e\theta} \end{bmatrix}$ is not simply the transpose of $\begin{bmatrix} K_{ee} & K_{e\theta} \\ 0 & K_{e\theta} \end{bmatrix}$ since $K_{ee}$ is not symmetric and since $K_{e\theta}$ is not the transpose of $K_{ee}$. Also a particular attention must be paid to the structure of the mass, damping and stiffness matrices in order to implement an efficient reduction algorithm. Indeed, it is preferable to factorize both $\begin{bmatrix} K_{ee} & K_{e\theta} \\ 0 & K_{e\theta} \end{bmatrix}$ and $\begin{bmatrix} K_{ee} & K_{e\theta} \\ 0 & K_{e\theta} \end{bmatrix}$ rather than factorize the entire $K_{ee}$. matrix.

It must also be noted that despite the symmetry of the mass matrix in the complete model, the reduced mass matrix of a thermo-mechanical system is non-symmetric. This comes from the fact that the test and trial spaces are not identical.

The choice for the internal modes could be based on the modes and Krylov vectors of the state-space form of the internal problem, similarly to what was proposed in [5, 6] for highly damped systems. In this work however we are not dealing with structural damping although the matrix equation looks similar to a structurally damped system. Here however the problem is typically composed of a structural and a thermal part that have well separate time constants, namely the period of the lowest pure structural frequencies is much smaller than the time constant of the thermal diffusion in the system.

Hence, for what concerns the thermal diffusion, a quasi-static behavior of the structure can be assumed as a good approximation. Therefore, the dynamic coupling between thermal and structural problem can be neglected.
in the process of internal mode computation. This last one is thus performed by solving two decoupled eigen
problems.

The structural symmetric eigen problem characterizes the dynamic response of the system to structural time
dependent loads.

$$K_{cc}^{uu} \Phi_u = M_{cc}^{uu} \Phi_u \Omega_u$$
\[
\begin{align*}
(\Phi_u)^T M_{cc}^{uu} \Phi_u &= I \\
(\Phi_u)^T K_{cc}^{uu} \Phi_u &= \Omega_u
\end{align*}
\]

Equation 16: Structural eigen problem

The “thermal driven” internal modes are computed in two steps. First, the decoupled thermal eigen problem is
solved. It is symmetric and represents the thermal response to thermal loadings:

$$K_{cc}^{\theta\theta} \Phi_\theta = D_{cc}^{\theta\theta} \Phi_\theta \Omega_\theta$$
\[
\begin{align*}
(\Phi_\theta)^T D_{cc}^{\theta\theta} \Phi_\theta &= I \\
(\Phi_\theta)^T K_{cc}^{\theta\theta} \Phi_\theta &= \Omega_\theta
\end{align*}
\]

Equation 17: Thermal eigen problem

Nevertheless, by considering separately the two sets of internal modes, respectively solution of the mechanical
and thermal problem, the coupling between the two physics is not considered, leading to poor quality
approximations.

In accordance with the assumption of quasi-static behaviour of the structure with respect of the thermal
excitations, a significant correction is proposed.

This approach, which is different from the current practice, consists in adding a coupling term to the thermal
internal modes that corresponds to the static structural response to the thermal eigen modes. For that we
consider the quasi-static structural problem

$$\begin{bmatrix} K_{cc}^{uu} & K_{cc}^{\theta\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_c \\ \theta_c \end{bmatrix} = 0$$

Equation 18

which can be solved for the structural part assuming that the thermal DOF are given in the thermal reduction
basis. Hence every thermal mode will be associated to a quasi-static structural response

$$\Phi_\theta = -(K_{cc}^{uu})^{-1} K_{cc}^{\theta\theta} \Phi_\theta$$

Equation 19

According to this quasi-static approximation, the trial and test basis are thus defined as follow:

$$q = \begin{bmatrix} u_r \\ \theta_r \\ u_c \\ \theta_c \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ K_{cc}^{uu} & K_{cc}^{\theta\theta} & \Phi_u & \Phi_\theta \\ 0 & K_{cc}^{\theta\theta} & 0 & \Phi_\theta \end{bmatrix} \begin{bmatrix} u_r \\ \theta_r \\ \gamma_u \\ \gamma_\theta \end{bmatrix} = R\hat{q}$$

Equation 20
\[
\tilde{R}^T = \begin{bmatrix}
    I & 0 & -K^{uu}_{rr} & K^{u\theta}_{rr} & K^{uu}_{cc} & K^{u\theta}_{cc} \\
    0 & I & 0 & K^{\theta\theta}_{rr} & 0 & K^{\theta\theta}_{cc} \\
    0 & 0 & \Phi_u^T & 0 & \Phi_{\theta}^T \\
    0 & 0 & 0 & \Phi_{u}^T & \Phi_{\theta}^T
\end{bmatrix}^{-1}
\]

**Equation 21**

## 6. Application

The efficiency of this approach has been evaluated using a simple model of a bi-layer cantilever beam, as illustrated on the following picture. The plate, made of two layers of different materials, is clamped at one side, free at the second and an iso-temperature condition is imposed on both faces. The displacements on the free end are constrained to move as one rigid face.

A surface heat source is applied on a quarter of the top face and, due to the difference of thermal expansion coefficient of the two materials, an out-of-plane displacement of the beam is induced. The model is reduced in terms of two thermal DOF, respectively the temperature of the clamped and free faces, and three mechanical DOF corresponding to the displacement of the free face centre point. Figure 1 describes the model and gives the static solution.

**Figure 1: Thermo-mechanical test model and static solution of thermal problem.**

The projection modes used to reduce the model are presented hereafter. The left figures show the temperature distribution while le right figures correspond to mechanical deformation.

The first three modes correspond to the retained mechanical DOF modes (“static modes”).

**Figure 2: Thermo mechanical projection mode related to X displacement DOF**
The two next modes correspond to the retained thermal DOF modes.
Additional internal modes are thermal eigen modes of the model constrained to get null value of retained DOF. Each of these thermal modes is updated by the addition of the corresponding mechanical quasi-static response, as illustrated hereafter.

Figure 7: Thermo mechanical internal projection mode 1

Figure 8: Thermo mechanical internal projection mode 2

Figure 9: Thermo mechanical internal projection mode 3

Figure 10: Thermo mechanical internal projection mode 4
No dynamic structural modes were included.
A transient simulation has been performed in which the DOF: the clamped face was constrained to reference temperature and the surface heat was applied according to a step function.
The following pictures compare the out-of-plane displacement and temperature of the free face centre point, computed using the full 3D model and using the reduced models generated using pure internal thermal modes and the improved “thermal driven” modes proposed in this paper.

Figure 13: Thermo mechanical transient response computed using the full model and the reduced model with and without the structural quasi-static correction (temperature plot and zoom around initial time).
Figure 14: Thermo mechanical transient response computed using the full model and the reduced model with and without the structural quasi-static correction (out-of-plane response).

From the response curves it is clearly that adding the quasi-static structural response to the thermal modes significantly improves the accuracy of the reduced model.

7. Conclusion

In this paper we have proposed an enrichment of the reduction basis allowing a more accurate approximation of the thermo-mechanical response. The idea consists in adding to the pure thermal modes the quasi-static structural response to the thermal modes. It has been shown on a simple cantilever bi-material beam example that the quasi-static structural enrichment of the thermal modes significantly improves the fidelity of the reduced model.

Being able to properly represent the multiphysical response using low order models is especially important in space applications. Therefore the proposed method has been implemented in a concurrent design environment as used at the Technical Center of the European Space Agency (the MS-Excel® environment) in order to show the feasibility of using reduced models at different stages of the design process.

References