Polyreference Frequency-Domain Least-Squares Estimation with Confidence Intervals

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ABSTRACT

The PolyMAX estimator is used intensively in modal analysis applications nowadays. The main advantages are its speed and the very clear stabilization diagrams it yields. Recently, an algorithm allowing a fast calculation of confidence intervals has been derived for frequency-domain least-squares estimators based on a common-denominator model. In this contribution, the approach is extended to the PolyMAX estimator, i.e. a polyreference frequency-domain least-squares estimator, based on a right matrix-fraction description. If the coherences of the measured frequency response functions (FRFs) are available, the covariance matrix of the estimated model parameters can be obtained without major additional calculations. The confidence intervals can then be calculated in a second step. The correctness of the approach is verified by means of Monte Carlo simulations.

NOMENCLATURE

- $H_m(\Omega_f)$: FRF right matrix-fraction model
- $H(\Omega_f)$: measured FRF
- $N(\Omega_f)$: numerator matrix polynomial
- $D(\Omega_f)$: denominator matrix polynomial
- $N_o(\Omega_f)$: Numerator matrix polynomial for output $o$
- $\Omega_f$: polynomial basis function
- $\theta$: coefficients matrix
- $\theta_D$: denominator coefficients matrix
- $\theta_N$: numerator coefficients matrix
- $\epsilon(\Omega_f, \theta, H)$: error equation
- $W(\omega_f)$: weighting matrix in error equation
- $J$: jacobian matrix
- $\Gamma_o$: jacobian submatrix corresponding to numerator coefficients
1 INTRODUCTION

The PolyMAX estimator [1], a non-iterative polyreference least-squares complex frequency-domain (LSCF) estimator, is one of the state-of-the-art modal parameter estimators. The PolyMAX has many advantages: it is polyreference, fast and yields very clear stabilization diagrams. One drawback is that it does not yield confidence intervals on the modal estimates, a property that the maximum likelihood (ML) estimator does have [2,3]. Another advantage of the ML estimator is that it is consistent. However, this iterative estimator is slower than the PolyMAX estimator and is not suited to generate stabilization diagrams.

The present paper proposes a method to derive the covariance matrix of the estimated coefficients for the PolyMAX estimator without major additional calculations. To this end, some mild assumptions and approximations have to be made. The approach assumes that the noise present on the frequency response functions is circular complex normally distributed and uncorrelated over the frequencies. The noise variances are assumed to be known. These variances can be easily estimated using the coherence functions [4].

Unlike the single-reference case [5], only a compact form of the covariance matrix of the denominator coefficients is obtained. The calculation of the full covariance matrix is evaded to conserve computational efficiency. The variances on the poles, frequencies and damping ratios can be calculated in a second step [6,7]. This problem will not be tackled in this paper.

The approach is also extended to other least-squares frequency-domain modal parameter estimators, e.g. the iterative quadratic maximum likelihood (IQML) estimator [8,9]. This estimator combines the nice properties of the PolyMAX estimator but has improved estimates at the price of iterating. It is shown that for the IQML estimator, the expression for the covariance matrix of the coefficients equals the expression for the ML estimator for large signal-to-noise ratios.

One specific field of application of the PolyMAX estimator with confidence intervals is flight flutter testing [10]. Flutter is a destructive aero-elastic instability of surfaces exposed to wind, e.g. aircraft wings. Current flight tests excite the wing surface in flight and estimate the frequency and damping ratio for a specific flight speed and height. If the vibration is stable, the aircraft proceeds to the next flight test point. This approach is very time consuming and thus expensive and dangerous. The damping varies extremely fast if an instability is encountered. Confidence intervals on the damping estimates enhance the safety of the flight test.

The covariance matrix can also be used in the construction of stabilization diagrams. Because of the noise present in the data, the
model order has to be chosen high enough to account for unmodelled dynamics. Due to this overmodelling however, also mathematical noise poles are estimated. It is expected that these mathematical poles will have a larger variance than the physical ones. It is possible to apply this approach even if no noise information on the FRFs is available. The variance on the poles will then have no physical meaning but the magnitude will still differ for physical poles compared to mathematical poles.

In Section 2, a brief review is given of the fast implementation of the PolyMAX estimator. Section 3 elaborates on the calculation of the covariance matrix of the estimated denominator coefficients. The Monte Carlo simulations results are presented in Section 4.

## 2 FREQUENCY DOMAIN POLYREFERENCE SYSTEM IDENTIFICATION

The PolyMAX estimator uses a right matrix-fraction description \[1] to model the FRF \( H_w(\Omega_f) \in \mathbb{C}^{N_w \times N_i} \) for every frequency line \( f \) as

\[
H_w(\Omega_f) = N(\Omega_f)D^{-1}(\Omega_f)
\]

with \( N(\Omega_f) \in \mathbb{C}^{N_w \times N_f} \) the numerator matrix polynomial and \( D(\Omega_f) \in \mathbb{C}^{N_f \times N_f} \) the denominator matrix polynomial, defined by

\[
N(\Omega_f) = \sum_{j=0}^{nN} N_j \Omega_f^j \\
D(\Omega_f) = \sum_{l=0}^{nN} D_l \Omega_f^l
\]

The matrix coefficients \( N_j \) and \( D_l \) are the parameters to be estimated. These coefficients are grouped together in one parameter matrix \( \theta \). The coefficients can be real \((nN_f = 2n_w)\) or complex \((nN_f = n_w)\). The polynomial basis function \( \Omega_f \) can be expressed in the continuous frequency domain \( (\Omega_f = i\omega_f) \) or in the discrete-time domain, where \( \Omega_f \) is given by \( \exp(-i\omega_f T_s) \) (z-domain) with \( T_s \) the sampling period. Also orthogonal polynomials are frequently used. In this paper, \( \theta \) is considered real and the z-domain formulation is used.

By replacing the model \( H_w(\Omega_f) \) by the measured FRF \( H(\omega_f) \) and right multiplying with the denominator polynomial \( D(\Omega_f) \), the linearized (weighted) error equation can be found as

\[
\varepsilon(\omega_f, \theta, H) = W(\omega_f)\left( N(\Omega_f) - H(\omega_f) D(\Omega_f) \right)
\]

\( W(\omega_f) \) is a diagonal frequency-dependent weighting matrix. Eq. (3) can be rewritten for all frequency lines \( f \) and for complex \( \theta \) as

\[
\varepsilon = J\theta = \begin{bmatrix} \Gamma_1 & 0 & \cdots & 0 \\
0 & \Gamma_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \Gamma_{N_f} \end{bmatrix} \begin{bmatrix} \chi_1 \\
\vdots \\
\vdots \\
\chi_{N_f} \end{bmatrix} = 0
\]

with

\[
\chi_f = \begin{bmatrix} \chi_{\Omega_f}^r(\Omega_{f_1}) \\
\chi_{\Omega_f}^r(\Omega_{f_2}) \\
\vdots \\
\chi_{\Omega_f}^r(\Omega_{f_{N_f}}) \end{bmatrix} \quad \chi_f = \begin{bmatrix} \chi_{\Omega_f}^i(\Omega_{f_1}) \\
\chi_{\Omega_f}^i(\Omega_{f_2}) \\
\vdots \\
\chi_{\Omega_f}^i(\Omega_{f_{N_f}}) \end{bmatrix}
\]

\[
\chi_f = \begin{bmatrix} \chi_{\Omega_f}^r(\Omega_{f_1}) \\
\chi_{\Omega_f}^r(\Omega_{f_2}) \\
\vdots \\
\chi_{\Omega_f}^r(\Omega_{f_{N_f}}) \\
\chi_{\Omega_f}^i(\Omega_{f_1}) \\
\chi_{\Omega_f}^i(\Omega_{f_2}) \\
\vdots \\
\chi_{\Omega_f}^i(\Omega_{f_{N_f}}) \end{bmatrix} = \begin{bmatrix} \theta_{N_1} \\
\vdots \\
\theta_{N_{N_f}} \end{bmatrix} \quad \theta_f = \begin{bmatrix} D_0 \\
D_1 \\
\vdots \\
D_{N_f} \end{bmatrix}
\]

where

\[
\chi_f = W_f(\omega_f)(\Omega_{f_1}, \Omega_{f_2}, \ldots, \Omega_{f_{N_f}})
\]

and

\[
\chi_f = -W_f(\omega_f)(\Omega_{f_1}, \Omega_{f_2}, \ldots, \Omega_{f_{N_f}}) \otimes H_w(\omega_f)
\]

\( \text{Re}.() \) and \( \text{Im}.() \) denote taking the real and imaginary part respectively. Note that the Jacobian matrix \( J \) is independent of the parameters to be estimated. \( J \) has size \( N_w N_f \times (n + 1)(N_w + N_i) \) and \( \theta \) \((n + 1)(N_w + N_i) \times N_f \). The number of frequencies can be eliminated from the dimensions of \( J \) by formulating the so-called normal equations

\[
J^T J \theta = 0
\]
with

\[
J^T J = \begin{bmatrix}
R_1 & \cdots & 0 & S_1 \\
\vdots & & \vdots & \vdots \\
0 & R_{N_0} & S_{N_0} \\
S_1^T & \cdots & S_{N_0}^T & \sum_{o=1}^{N_0} T_o
\end{bmatrix}
\]

(10)

with \( R_o = \Gamma_o^T T_o \), \( S_o = \Gamma_o^T T_o \), and \( T_o = T_o^T T_o \) the transpose. The \( R_o, S_o \) and \( T_o \) submatrices have a toeplitz structure enabling a fast construction (4)\(^{[12]}\).

\[
[R_o]_{ij} = \sum_{f=1}^{N_f} W_o(\omega_f)^2 \Omega_f^{i-j}
\]

\[
[S_o]_{ij} = \sum_{f=1}^{N_f} W_o(\omega_f)^2 H_o(\omega_f) \Omega_f^{i-j}
\]

\[
[T_o]_{ij} = \sum_{f=1}^{N_f} W_o(\omega_f)^2 H_o^T(\omega_f) H_o(\omega_f) \Omega_f^{i-j}
\]

with \( i = [(r-1)N_i + 1 : rN_i] \), \( j = [(s-1)N_i + 1 : sN_i] \) for both \( r, s = 1, 2, \ldots, n + 1 \).

Elimination of the numerator coefficients from Eq. (9) by substitution of

\[
\theta_{N_0} = -R_o^{-1} S_o \theta_D
\]

(12)

results in the so-called reduced normal equations

\[
\left[ \sum_{o=1}^{N_0} (T_o - S_o^T R_o^{-1} S_o) \right] \theta_D = M \theta_D = 0
\]

(13)

with \( M \) a square \((n + 1)N_i \times (n + 1)N_i \) matrix which is much smaller than the original normal matrix in Eq. (10). The least-squares (LS) solution of Eq. (13) is found by fixing one of the denominator coefficients, e.g. the one corresponding to the highest order coefficient, to the identity matrix \( I_{N_i} \)

\[
\theta_D = \left[ -M(1 : nN_i, 1 : nN_i)^{-1} [M(1 : nN_i, nN_i + 1 : (n + 1)N_i)] \right]^{-1}
\]

(14)

It has been shown that the choice of the constrained coefficient greatly influences the clarity of the stabilization diagram\(^{[13]}\). Once the denominator coefficients are obtained, back substitution in Eq. (12) is used to find the numerator coefficients.

The poles of the denominator matrix polynomial \( D \) are determined by solving an eigenvalue decomposition of the companion matrix \( D_{com} \), given by

\[
D_{com} = \begin{bmatrix}
D_o^{n-1} & \cdots & D_o & D_o
\end{bmatrix}
\]

(15)

with \( D_o = -D_o^{-1} D_o^T \). The \( r \)th modal participation vector \( L_r \) is related to the eigenvector \( V_r \) according to

\[
V_r = \begin{bmatrix}
\lambda_r^{n-1} L_r \\
\vdots \\
\lambda_r L_r \\
L_r
\end{bmatrix}
\]

(16)

As the denominator coefficients were estimated in the discrete time domain, the poles \( \lambda_r \) must still be converted to the continuous time domain.

As a last step, the mode shapes are calculated or estimated using the Least-Squares Frequency-Domain (LSFD) estimator. This last approach has the advantage that also upper and lower residuals can be computed\(^{[12]}\).
3 UNCERTAINTY CALCULATION

As the final goal is to calculate the variances on the poles and thus on the damping ratios and frequencies, only the covariance matrix of the denominator coefficients is needed. A computation-time friendly approach analogous to the estimation algorithm of the PolyMAX estimator is used to speed up the calculation of the covariance matrix of the denominator coefficients.

3.1 Perturbation on estimated coefficients

Starting from Eq. (13), an expression for the perturbation on \( \theta_D \) can be calculated. Say \( \hat{\theta}_D \) is the exact solution of the normal equations Eq. (13) in the noiseless case

\[ M \hat{\theta}_D = 0 \]  
\[ (17) \]

If a perturbation \( \Delta H \) is applied on the measured FRF and thus on \( M \), the solution will differ by an amount \( \Delta \theta_D \)

\[ (M + \Delta M)(\hat{\theta}_D + \Delta \theta_D) = 0 \]
\[ (18) \]

If the higher order terms are neglected, Eq. (18) can be reduced to

\[ M \Delta \theta_D = -\Delta M \hat{\theta}_D \]  
\[ (19) \]

The goal is to find an expression for \( \Delta \theta_D \) as function of the perturbation \( \Delta H \) on the FRF. Using Eq. (13)

\[ M = \sum_{o=1}^{N} \left( T_o - S^T_o R_o^T S_o \right) \]
\[ = \sum_{o=1}^{N} \left( \Gamma_o^T \Upsilon_o \right) \]  
\[ (20) \]

an expression is found for \( \Delta M \)

\[ \Delta M = \sum_{o=1}^{N} \left( \Delta \Upsilon_o + \Delta \Gamma_o^T \Upsilon_o = \Delta \Gamma_o^T \Upsilon_o \right) \]  
\[ (21) \]

It is noted that the perturbation on \( M \) is caused by a perturbation on the FRF \( H \) and thus only affects the \( \Upsilon_o \) submatrices. Eq. (19) can now be rewritten as

\[ M \Delta \theta_D = -\left[ \sum_{o=1}^{N} \left( \Xi_o \Delta \Upsilon_o + \Delta \Gamma_o^T \Upsilon_o \right) \right] \hat{\theta}_D \]
\[ (22) \]

with \( \Xi_o = \Upsilon_o - \Gamma_o (\Upsilon_o \Gamma_o)^{-1} \Upsilon_o \). The second term in Eq. (22) equals 0 as

\[ \Xi_o \hat{\theta}_D = \Upsilon_o \hat{\theta}_D - \Gamma_o (\Upsilon_o \Gamma_o)^{-1} \Upsilon_o \hat{\theta}_D \]
\[ = \Upsilon_o \hat{\theta}_D - \Gamma_o R_o^{-1} S_o \hat{\theta}_D \]
\[ = \Upsilon_o \hat{\theta}_D + \Gamma_o S_o = 0 \]  
\[ (23) \]

Thus \( \Delta \theta_D \) is given by

\[ \Delta \theta_D = -M^{-1} \sum_{o=1}^{N} \left( \Xi_o \Delta \Upsilon_o \hat{\theta}_D \right) \]
\[ (24) \]

3.2 Compact formulation of covariance of denominator coefficients

The full covariance matrix is given by

\[ \text{Cov}(\theta_D) = E \left\{ \text{vec}(\Delta \theta_D) \text{vec}(\Delta \theta_D)^T \right\} \]
\[ (25) \]

with \( \text{vec}(.) \) the column stacking operator. The use of the \( \text{vec} \) operator is avoided as this blows up the matrix size with the number of inputs \( N_i \) and slows down the calculation. A compact formulation of the covariance matrix of the denominator coefficients can be found as

\[ E \left[ \Delta \theta_D \Delta \theta_D^T \right] \]
\[ (26) \]
The compact formulation is calculated using Eq.(24)

\[ E \left[ \Delta \mathbf{\theta}_0 \Delta \mathbf{\theta}_0^T \right] = E \left\{ M^{-1} \left[ \sum_{n=1}^{N_o} \left( \mathbf{\Xi}_n \Delta \Psi_{n0} \Delta \Psi_{n0}^T \Xi_n \right) \right] M^{-1} \right\} \]

\[ = M^{-1} \left[ \sum_{n=1}^{N_o} \left( \mathbf{\Xi}_n E \left[ \Delta \Psi_{n0} \Delta \Psi_{n0}^T \right] \Xi_n \right) \right] M^{-1} \] \hfill (27)

Using Eq.(4), and assuming that the weighting in Eq.(3) is equal to the identity matrix, it is easily found that

\[ \Delta \Psi_{n0} = \begin{bmatrix} -[\Delta H_n(\omega_1)] \left[ \mathbf{D}(\Omega_1) \right] \\ -[\Delta H_n(\omega_2)] \left[ \mathbf{D}(\Omega_2) \right] \\ \vdots \\ -[\Delta H_n(\omega_N)] \left[ \mathbf{D}(\Omega_N) \right] \end{bmatrix} \] \hfill (28)

Under the assumption that there is no correlation over the frequencies, \( E \left[ \Delta \Psi_{n0} \Delta \Psi_{n0}^T \right] \) is a \( N_f \times N_f \) matrix \( \mathbf{P}_{a,k} \) with entries

\[ (\mathbf{P}_{a,k})_{f,g} = E \left[ \Delta H_n(\omega_f) \mathbf{D}(\Omega_f) \mathbf{D}(\Omega_g)^T \Delta H_n(\omega_g) \right] \]

\[ = E \left[ \text{tr} \left( \Delta H_n(\omega_f) \mathbf{D}(\Omega_f) \mathbf{D}(\Omega_g)^T \Delta H_n(\omega_g) \right) \right] \]

\[ = \text{tr} \left( \mathbf{D}(\Omega_f) \mathbf{D}(\Omega_g)^T E \left[ \Delta H_n(\omega_f) \Delta H_n(\omega_g) \right] \right) \delta_{fg} \] \hfill (29)

with \( \delta_{fg} \) the kronecker delta. Thus \( \mathbf{P}_{a,k} \) is a diagonal matrix with entries

\[ (\mathbf{P}_{a,k})_{f,f} = \text{tr} \left( \mathbf{D}(\Omega_f) \mathbf{D}(\Omega_f)^T \text{Cov} \left( \mathbf{H}_n(\omega_f), \mathbf{H}_n(\omega_f) \right) \right) \] \hfill (30)

Eq.(27) can now be written as

\[ E \left[ \Delta \mathbf{\theta}_0 \Delta \mathbf{\theta}_0^T \right] = M^{-1} \left[ \sum_{n=1}^{N_o} \left( \mathbf{\Xi}_n^T \mathbf{P}_{a,k} \mathbf{\Xi}_n \right) \right] M^{-1} \] \hfill (31)

Elaborating \( \mathbf{\Xi}_n^T \mathbf{P}_{a,k} \mathbf{\Xi}_n \) gives

\[ \mathbf{\Xi}_n^T \mathbf{P}_{a,k} \mathbf{\Xi}_n = \left[ \mathbf{\Gamma}_o^T - \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \left( \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \right)^{-1} \mathbf{\Gamma}_o^T \mathbf{Y}_k \right] \mathbf{P}_{a,k} \left[ \mathbf{\Gamma}_o^T - \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \left( \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \right)^{-1} \mathbf{\Gamma}_o^T \mathbf{Y}_k \right] \]

\[ = \mathbf{\Gamma}_o^T \mathbf{P}_{a,k} \mathbf{\Gamma}_o - \mathbf{\Gamma}_o^T \mathbf{P}_{a,k} \mathbf{\Gamma}_o \left( \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \right)^{-1} \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \mathbf{P}_{a,k} \mathbf{\Gamma}_o + \mathbf{\Gamma}_o^T P_{a,k} \left( \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \right)^{-1} \mathbf{\Gamma}_o^T \mathbf{Y}_k \]

\[ + \mathbf{\Gamma}_o^T \mathbf{P}_{a,k} \mathbf{\Gamma}_o \left( \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \right)^{-1} \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \mathbf{P}_{a,k} \mathbf{\Gamma}_o + \mathbf{\Gamma}_o^T \mathbf{P}_{a,k} \mathbf{\Gamma}_o \left( \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \right)^{-1} \mathbf{\Gamma}_o^T \mathbf{Y}_k \] \hfill (32)

The compact formulation of the covariance matrix can thus be calculated as

\[ E \left[ \Delta \mathbf{\theta}_0 \Delta \mathbf{\theta}_0^T \right] = M^{-1} \left[ \sum_{n=1}^{N_o} \left( \mathbf{T}_{P_{a,k}} - \mathbf{S}_o^T \mathbf{R}_o^{-1} \mathbf{S}_o \right) \right] M^{-1} \] \hfill (33)

with

\[ \mathbf{R}_o = \mathbf{\Gamma}_o^T \mathbf{\Gamma}_o \]

\[ \mathbf{S}_o = \mathbf{\Gamma}_o^T \mathbf{\theta}_o \]

\[ \mathbf{T}_o = \mathbf{\Gamma}_o^T \mathbf{\theta}_o \]

\[ \mathbf{R}_{P_{a,k}} = \mathbf{\Gamma}_o^T \mathbf{P}_{a,k} \mathbf{\Gamma}_o \]

\[ \mathbf{S}_{P_{a,k}} = \mathbf{\Gamma}_o^T \mathbf{P}_{a,k} \mathbf{\theta}_o \]

\[ \mathbf{T}_{P_{a,k}} = \mathbf{\Gamma}_o^T \mathbf{P}_{a,k} \mathbf{\theta}_o \] \hfill (34)

3.3 Some further elaboration under specific assumptions

If it is assumed that also the noise on the outputs \( a,k \) is uncorrelated, \( (\mathbf{P}_{a,k})_{f,f} = (\mathbf{P}_{a,k})_{f,f} \delta_{a,k} \). Eq.(33) is now reduced to only one summation over the outputs

\[ E \left[ \Delta \mathbf{\theta}_0 \Delta \mathbf{\theta}_0^T \right] = M^{-1} \left[ \sum_{n=1}^{N_o} \left( \mathbf{T}_{P_o} - \mathbf{S}_o^T \mathbf{R}_o^{-1} \mathbf{S}_o \right) \right] M^{-1} \] \hfill (35)
E.g. for laser measurements every output is measured separately (i.e. at different times) prohibiting correlation. This assumption is somewhat more severe than to assume that the noise is not correlated over the frequencies, which is generally the case.

A further significant reduction of the calculation of the covariance matrix of the denominator coefficients can be obtained by choosing a specific weighting in Eq. (3). The Iterative Quadratic Maximum Likelihood (IQML) estimator\cite{8,9} is an iterative least-squares estimator with weighting

$$W_m(\omega_f) = \text{tr} \left( D_m^{-1}(\Omega_f)D_m^{-1}(\Omega_f)^T \text{Cov} \left( H_{o}(\omega_f) \right) \right)^{-1/2}$$

an estimate of the ‘optimal’ maximum-likelihood (ML) weighting. \((\cdot)^m\) denotes the \(m\)th iteration step. As for this weighting the knowledge of the denominator matrix polynomial is necessary, an iterative approach is needed. Every iteration step, the weighting is updated using the calculated denominator of the previous step. It can be shown for sufficiently high signal-to-noise ratios that the cost function of the IQML converges to the ML cost function\cite{8}. However, it cannot be shown that the estimator is consistent.

If the weighting \(W_m(\omega_f)\) is chosen as in Eq. (36), the diagonal weighting \(P_o\) in Eq. (34) equals the identity matrix \(I_{N_f}\). The expression for the covariance matrix Eq. (35) now reduces to

$$E \left[ \Delta \theta_n \Delta \theta_D^T \right] = M^{-1} \sum_{t=1}^{N_u} \left( T_o - S_o^T R_o^{-1} S_o - S_o^T R_o^{-1} R_o S_o + S_o^T R_o^{-1} R_o R_o^{-1} S_o \right) M^{-1}$$

$$= M^{-1} \sum_{t=1}^{N_u} \left( T_o - S_o^T R_o^{-1} S_o \right) M^{-1}$$

$$= M^{-1} M M^{-1}$$

$$= M^{-1}$$

It is thus sufficient to invert the reduced normal equations matrix \(M\) obtained in the last iteration step to calculate the covariance matrix on the denominator coefficients. Note that Eq. (40) is identical to the expression of the ML estimator for sufficiently high signal-to-noise ratios.

The derivation of the covariance matrix of the coefficients starts from the knowledge of the covariance matrix of the measured FRFs. These can be easily derived if the coherence functions are available\cite{14,14} using

$$\text{Cov} \left( H_{o}(\omega_f) \right) = E \left[ \Delta H_o^T(\omega_f) \Delta H_o(\omega_f) \right] = \frac{1}{M} (1 - \gamma_o^2(\omega_f)) S_Y S_F^{-1}$$

with \(\gamma_o^2\) the multiple coherence function, \(S_Y, S_F\) the autopower spectra of the measured responses \(Y_o\) and forces \(F\), and \(M\) the number of averages used in the non-parametric preprocessing step.

4 SIMULATIONS

Monte Carlo simulations were performed to assess the validity of Eq. (33). A system with three modes, three outputs and two inputs was generated in the frequency band 0.3 – 3 Hz. Circular complex constant noise uncorrelated over the outputs was added to the exact FRF and the variances were used to construct the covariance matrix of the simulated measurements. Fig. 1 shows the FRF for one output and one input. 1000 disturbed data sets were generated and estimated using the PolyMAX estimator with covariance computation. The covariance matrix resulting from the Monte Carlo simulations is compared to one estimated covariance matrix in Fig. 2.

It is clear that the structure of the covariance matrix is well estimated. The numerical accuracy can be verified in Table 1 where an excerpt of an estimated covariance matrix is compared to the covariance matrix obtained by averaging the estimated denominator coefficients from the Monte Carlos simulations. The order of magnitude of the Monte Carlo covariance matrix and the estimated one matches quite good. The largest deviation of a diagonal element is about 25%.

5 CONCLUSIONS

In this contribution, a method is proposed to calculate in a fast way the covariance matrix of estimated denominator coefficients for the PolyMAX estimator, i.e. a polyreference frequency-domain least-squares estimator. Using the specific structure of the submatrices of
Figure 1: FRF of system under test for one input-output combination: exact FRF (solid line), measured FRF (solid and dots) and noise level (dotted)

Figure 2: Covariance matrix: Monte Carlo averaged covariance matrix (left), sample estimated covariance matrix (right)

TABLE 1: Covariance matrix excerpt: Monte Carlo averaged covariance matrix (above line), sample estimated covariance matrix (below line)
the reduced normal equations, a computationally efficient algorithm is derived. The approach is also extended to the Iterative Quadratic Maximum Likelihood estimator. It is shown that for the IQML estimator the computational burden can be further reduced. This procedure indicates strong analogies to the Maximum Likelihood estimator. Monte Carlo simulations are used to validate the proposed algorithms.

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