Vibration Mode Shape Recognition Using the Zernike Moment Descriptor

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ABSTRACT

Vibration mode-shape comparison between numerical models and experimental data is an essential step in the study of structural finite element model (FEM) updating. The Modal Assurance Criterion (MAC) is the most popular method for such comparison at the moment, which works perfectly well for small and medium sized structures. MAC provides a measure of closeness between the predicted and measured eigenvectors but contains no explicit information on shape features. This is especially significant for large and complicated industrial systems where the MAC index can rapidly degenerate over the course of just a few modes so that a very detailed finite element model, which surely represents the physical system with good accuracy, cannot be reliably compared to measurements. New techniques, based upon the well-developed philosophies of image processing (IP) and pattern recognition (PR) are considered in this paper. The Zernike moment descriptor (ZMD) is a region-based shape descriptor having outstanding properties in IP including rotational invariance, expression and computing efficiency, ease of reconstruction and robustness to noise. In this paper the ZMD is applied to the problem of mode shape recognition for a circular plate. Result shows that the ZMD has considerable advantages over the traditional MAC index when identifying the cyclically symmetric mode shapes that occur in axisymmetric structures with repeated eigenvalues.

NOMENCLATURE

\[ m_{p,q}(\mu_{p,q}) \] Geometric Moments (Central)
\[ Z_{n,m} \] Zernike Moments
\[ V_{n,m}(x,y), V_{n,m}(\rho, \theta) \] Zernike Polynomials
\[ R_{n,m} \] Radial Polynomials
\[ I(x,y), I(\rho, \theta) \] Image or Shape Pattern
\[ x, y, \rho, \theta \] Cartesian and Polar Coordinates
\[ n, p, q \] Orders of Moments
\[ m \] Repetition of Zernike Moments
\[ \delta_{i,j} \] Kronecker Delta
\[ \Re(\ast), \Im(\ast) \] Real and Imaginary Part of Complex Number *
1. INTRODUCTION

Experimental validation is a crucial step in obtaining a reliable numerical model (e.g., a finite element model). Vibration mode-shapes are important properties in structural dynamics. The current method for comparing the predicted and measured mode shapes is the modal assurance criterion (MAC) of Allemang and Brown [1]. The MAC has been widely and successfully adopted in finite element model updating [2-5] for the last two decades because it provides a simple tool to assess the quality of numerical models. However, all the information on similarity between the finite-element prediction and test data is encapsulated in a single number. This makes the appreciation of subtle changes in shape, either locally or globally unrealistic by MAC. This becomes critical in the case of large and complicated structures with mode shapes that can be fully measured by modern scanning laser techniques and accurately predicted by detailed finite element models. A new technique is presented in this paper that allows a more detailed understanding of mode shapes to be obtained in the form of ‘shape features’.

Image processing (IP) and pattern recognition (PR) provide many methods for dealing with the shape description and comparison of shapes [6-9]. Within such procedures, a series of shape features with good discriminative capability should be extracted to form a feature vector – shape descriptor (SD). Now the similarity/dissimilarity of the shapes can be revealed by the ‘distance’ of their corresponding SDs in the shape feature space according to appropriate criteria.

The moment descriptor is one of the most popular shape descriptors in image processing and pattern recognition of two dimensional images and shape patterns [10, 11]. A set of moment invariants constructed by non-linear combinations of geometric moments was first introduced by Hu [12]. These geometric moment invariants have the properties of being invariant under image translation, scaling and rotation. However, the basis of geometric moments is not orthogonal, the moment sequence includes redundant information to a high degree and the high-order moments are very sensitive to noise. Furthermore, the reconstruction of the image from these moments is very difficult.

The idea of orthogonal moments to recapture the image from the moments based on the theory of orthogonal polynomials was suggested by Teague [13]. The Zernike moment (ZM) is based on a complete set of orthogonal polynomials over a circle of unit radius – Zernike polynomials. It was found later that the Zernike moment is one of the most important region-based shape descriptors because of its outstanding properties resulting from the orthogonality of the Zernike polynomials. Firstly, expressing an image as a set of mutually independent descriptors has the effect of reducing the information redundancy of the ZM to a minimum. Secondly, the contribution of each order of moment to the image reconstruction can be separated, so that the process of regaining the original image is much easier than by geometric moment descriptors [14]. Rotational invariance [15, 16] is another important property of ZM, meaning that rotating an image does not change the magnitudes of its Zernike moments. Also, the ZM is robust to noise [14] and effective, meaning that a small number of Zernike moments are usually sufficient for shape reconstruction.

Therefore, the Zernike moment descriptor (ZMD) is widely adopted in the image processing and pattern recognition community. It is especially useful for, but not limited to, the study of circular or spherical symmetry, for example in optometry and ophthalmology [17-19] it is suitable for characterizing the optical aberrations of the cornea. In this paper we include the problem of mode-shape recognition in axisymmetric structures.

The definition of the Zernike moment and its properties are presented in Section 2. The application of mode-shape recognition by the Zernike moment descriptor to simple plate structures such as circular and rectangular plates are demonstrated in Section 3. A complete description of the ZMD in mode-shape recognition and more applications including finite element model updating, based on the sensitivities of the Zernike moments to small perturbations in system parameters, are described by the present authors [20]. Also, details of further applications of other shape descriptors and classification techniques for vibration mode shape recognition may be found in the first author’s PhD thesis [21].
2. ZERNIKE MOMENTS

Geometric moment is one of the simplest shape descriptors, defined as,

\[ m_{p,q} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^p y^q I(x,y) \, dx \, dy \quad p, q = 0, 1, 2, \ldots \]  
(1)

where \( I(x, y) \) is the image (or shape) to be described. The central moment is defined as,

\[ \mu_{p,q} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{x})^p (y - \bar{y})^q I(x,y) \, dx \, dy \quad p, q = 0, 1, 2, \ldots \]  
(2)

where

\[ \bar{x} = \frac{m_{10}}{m_{00}}, \quad \bar{y} = \frac{m_{01}}{m_{00}} \]  
(3), (4)

Hu’s seven moment invariants, including rotation, scaling, translation and skew orthogonal invariants, were derived based on equations (1) and (2). However, the kernel functions \( x^p y^q \) or \( (x - \bar{x})^p (y - \bar{y})^q \) are not orthogonal bases, the resulted geometric moment descriptors generally contain redundant information, are very sensitive to noise and shape reconstruction is difficult [13]. Therefore, moment descriptors derived from orthogonal kernel functions are considered. The Zernike moment descriptor is one of the best choices.

2.1 Definition

The Zernike moment of an image \( I(x, y) \) can be defined as,

\[ Z_{n,m} = \frac{n + 1}{\pi} \int_{x^2 + y^2 \leq 1} I(x, y)V_{n,m}^*(x, y) \, dx \, dy \]  
(5)

or expressed in a polar coordinate system as,

\[ Z_{n,m} = \frac{n + 1}{\pi} \int_0^{2\pi} \int_0^1 I(\rho, \theta)V_{n,m}^*(\rho, \theta) \rho d\rho d\theta \]  
(6)

where * denotes the complex conjugate and \( V_{n,m}(x, y) \) or \( V_{n,m}(\rho, \theta) \) is the complex Zernike polynomial introduced by Zernike [19] and can be expressed as,

\[ V_{n,m}(x, y) = V_{n,m}(\rho, \theta) = R_{n,m}(\rho) e^{im\theta} \]  
(7)

where

\[ i = \sqrt{-1} \]

\( n \) Non-negative integer, representing the order of the radial polynomial;

\( m \) Positive and negative integers subject to constraints \( n - |m| \) even, \( |m| \leq n \) representing the repetition of the azimuthal angle;

\( \rho \) Length of vector from the origin to \( (x, y) \);

\( \theta \) The azimuthal angle between vector \( \rho \) and the \( x \)-axis in counter-clockwise direction and

\( R_{n,m} \) Radial polynomial is defined as

\[ R_{n,m}(\rho) = \sum_{s=0}^{n-|m|/2} (-1)^s \frac{1}{s!} \frac{(n-s)!}{\left(\frac{n+|m|}{2} - s\right)! \left(\frac{n-|m|}{2} - s\right)!} \rho^{n-2s} \]  
(8)

Complex Zernike polynomials satisfy the orthogonality condition [22],
\[ \iint_{x^2+y^2\leq 1} V_{pq}(x,y)V_{nm}^*(x,y)\,dx\,dy = \frac{\pi}{n+1} \delta_{n,p} \delta_{m,q} \]  
(9)

where and \( \delta_{n,p} \) and \( \delta_{m,q} \) are Kronecker deltas.

For a digital image, the Zernike moments may be calculated by replacing the integral in equation (5) with the summation,

\[ Z_{n,m} = \frac{n+1}{\pi} \sum_{x} \sum_{y} I(x,y)V_{nm}^*(x,y) \quad x^2 + y^2 \leq 1 \]  
(10)

2.2 Reconstruction

According to the completeness and orthogonality of the Zernike polynomials [23], the original image \( I(\rho, \theta) \) may easily be reconstructed as,

\[ I(\rho, \theta) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} Z_{n,m} V_{nm}(\rho, \theta) \]  
(11)

where \( m \) is constrained as in equation (7). However, approximation of the reconstructed image by truncating the higher orders of Zernike moments must be carried out to reduce the computational demand in real applications. Therefore, the finitely reconstructed image \( \tilde{I}(\rho, \theta) \), which is obtained by keeping the moments from order 0 to \( N_{max} \) and discarding the remaining higher order Zernike polynomials, may be written as,

\[ \tilde{I}(\rho, \theta) = \sum_{n=0}^{N_{max}} \sum_{m=-n}^{n} Z_{n,m} V_{nm}(\rho, \theta) \]  
(12)

Furthermore, the right hand side of equation (12) can be expressed with the real-valued functions [16] as,

\[ \tilde{I}(\rho, \theta) = \sum_{n} \sum_{m>0} \left[ C_{n,m} \cos(m\theta) + S_{n,m} \sin(m\theta) \right] R_{n,m}(\rho) + \frac{C_{n,0}}{2} R_{n,0}(\rho) \]  
(13)

where

\[ C_{n,m} = 2\Re(Z_{n,m}) = \frac{2n+2}{\pi} \int_{0}^{2\pi} \int_{0}^{1} I(\rho, \theta) R_{n,m}(\rho) \cos(m\theta) \, \rho \, d\rho \, d\theta \]  
(14)

and

\[ S_{n,m} = -2\Im(Z_{n,m}) = \frac{-2n-2}{\pi} \int_{0}^{2\pi} \int_{0}^{1} I(\rho, \theta) R_{n,m}(\rho) \sin(m\theta) \, \rho \, d\rho \, d\theta \]  
(15)

Therefore, the reconstructed image \( \tilde{I}(\rho, \theta) \) calculated by equation (13) is guaranteed to be real-valued whatever the number \( N_{max} \) is selected. The sufficient large number of \( N_{max} \) can be determined by a predefined threshold when comparing the similarity between the original image \( I(\rho, \theta) \) and the approximation \( \tilde{I}(\rho, \theta) \). Correlation is one of the methods that can be applied for such comparison, defined as

\[ q(I, \tilde{I}) = \frac{\iint_{\Omega} (I - \tilde{I})(I - \tilde{I})\,dA}{\sqrt{\iint_{\Omega} (I - \tilde{I})^2\,dA \left[ \iint_{\Omega} (I - \tilde{I})^2\,dA \right]}} \]  
(16)

where \( \Omega \) denotes the internal domain of the unit disc, \( dA \) is the infinitesimal area, and
2.3 Rotational Invariance

If the image $I(\rho, \theta)$ is rotated by angle $\alpha$ with respect to the z-axis, we obtain

$$I'(\rho, \theta) = I(\rho, \theta - \alpha) \quad (19)$$

where $I'(\rho, \theta)$ denotes the rotated image. The Zernike moment of $I'(\rho, \theta)$ can be expressed as

$$Z_{n,m}^r = \frac{n + 1}{\pi} \int_0^{2\pi} \int_0^1 I(\rho, \theta - \alpha) R_{n,m} e^{-im\theta} \rho d\rho d\theta \quad (20)$$

$$= \frac{n + 1}{\pi} \int_0^{2\pi} \int_0^1 I(\rho, \theta - \alpha) R_{n,m} e^{-im(\theta - \alpha)} e^{-im\alpha} \rho d\rho d(\theta - \alpha) \quad (21)$$

$$= \left[ \frac{n + 1}{\pi} \int_0^{2\pi} \int_0^1 I(\rho, \theta - \alpha) R_{n,m} e^{-im(\theta - \alpha)} \rho d\rho d(\theta - \alpha) \right] e^{-im\alpha} \quad (22)$$

$$= Z_{n,m} e^{-im\alpha} \quad (23)$$

It is obvious that the amplitudes and phases of the both sides of equation (23) are equivalent. i.e.,

$$|Z_{n,m}^r| = |Z_{n,m}| \quad (24)$$

and

$$\arg(Z_{n,m}^r) = \arg(Z_{n,m}) - m\alpha \quad (25)$$

Therefore, the rotated image has the same amplitude of the Zernike moment as the original image by equation (24), and their angular difference can be determined by equation (25). It will be demonstrated later that these properties are useful for recognising mode shapes of axisymmetric structures.

3. CASE STUDY

Two example problems including a freely supported circular disc and a rectangular plate are presented in this section. The vibration mode shapes are determined and Zernike moments obtained for the both cases. Mode-shape correlations based on MAC and ZMD are compared. The results show the advantages of adopting the ZMD. Not only are the amplitudes of the ZMDs important but the angular orientation is shown to be significant especially in correlating the cyclically symmetric modes of axisymmetric structures.

3.1 Disc with Free Boundary

A finite element disc model is constructed as shown in Figure (1). The description of the model is given in Table 1. Normal modes of the disc were computed, with natural frequencies as shown in Table 2 and mode shape patterns shown in Figure 2. According to the conventional comparison method, the AutoMAC was calculated and shown in Figure 3. It is clear that almost none of the single modes has any correlation with others except itself. However, the disc is a perfectly axisymmetric structure and double modes are obtained. For instance, mode 2 has exactly the same shape as mode 1 but with a $45^\circ$ rotation with respect to z-axis. Similarly, there is $30^\circ$ rotational difference between modes 4 and 5; $90^\circ$ between modes 6 and 7, etc. Since the MAC does not indicate the double modes, the Zernike moment descriptors are considered.
Table 1: Finite Element Model Description of the Disc

<table>
<thead>
<tr>
<th>Element Type</th>
<th>4-Node Quadrilateral</th>
</tr>
</thead>
<tbody>
<tr>
<td>Element Property</td>
<td>Shell</td>
</tr>
<tr>
<td>Material</td>
<td>Steel</td>
</tr>
<tr>
<td>Boundary conditions</td>
<td>Free</td>
</tr>
<tr>
<td>Solution type</td>
<td>Normal Modes</td>
</tr>
<tr>
<td>Freq. Range</td>
<td>0 – 630 Hz</td>
</tr>
<tr>
<td>Number of Modes</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 2: Natural Frequencies of the Free-Free Disc

<table>
<thead>
<tr>
<th>Mode</th>
<th>Natural Frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>52.72</td>
</tr>
<tr>
<td>2</td>
<td>52.74</td>
</tr>
<tr>
<td>3</td>
<td>91.12</td>
</tr>
<tr>
<td>4</td>
<td>122.53</td>
</tr>
<tr>
<td>5</td>
<td>122.63</td>
</tr>
<tr>
<td>6</td>
<td>207.24</td>
</tr>
<tr>
<td>7</td>
<td>207.69</td>
</tr>
<tr>
<td>8</td>
<td>215.27</td>
</tr>
<tr>
<td>9</td>
<td>215.56</td>
</tr>
<tr>
<td>10</td>
<td>330.67</td>
</tr>
<tr>
<td>11</td>
<td>331.1</td>
</tr>
<tr>
<td>12</td>
<td>356.62</td>
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<tr>
<td>13</td>
<td>357.73</td>
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<tr>
<td>14</td>
<td>396.95</td>
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<tr>
<td>15</td>
<td>468.38</td>
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<td>16</td>
<td>469.37</td>
</tr>
<tr>
<td>17</td>
<td>535.64</td>
</tr>
<tr>
<td>18</td>
<td>537.38</td>
</tr>
<tr>
<td>19</td>
<td>627.69</td>
</tr>
<tr>
<td>20</td>
<td>628.8</td>
</tr>
</tbody>
</table>

Firstly, the necessary and sufficient order of the ZMD, $N_{max}$, needs to be determined to distinguish different shapes efficiently and to retrieve the original shapes accurately. $N_{max} = 8$ is adequate for the first 20 modes was determined by equation (16) and the results are shown in Figure 6. The amplitudes of the Zernike moments are shown in Figure 4. It is clear that just a few significant Zernike moments are sufficient to describe each mode. Especially for the first 18 modes, only one or two Zernike moments are sufficient to represent each separate mode shape with very good precision.

According to equation (25), the amplitudes of the Zernike moments of rotated images remain unchanged. The correlation coefficients of vectors having terms given by the Zernike-moment amplitudes $|Z_{n,m}^r|$ are shown in Figure 5.

$$
q \left( |Z_{n,m}|, |Z_{n,m}'| \right) = \frac{\sum_{k=1}^{K} (|Z_{n,m}| - |Z_{n,m}'|) (|Z_{n,m}'| - |Z_{n,m}|)}{\left[ \sum_{k=1}^{K} (|Z_{n,m}| - |Z_{n,m}'|)^2 \right]^{1/2} \left[ \sum_{k=1}^{K} (|Z_{n,m}'| - |Z_{n,m}|)^2 \right]^{1/2}}
$$

(26)

where $i$ and $j$ denote the mode number, $k$ denotes the $k^{th}$ Zernike moment and $K$ denotes the maximum number of Zernike moment considered, and $|Z_{n,m}|$ denotes the mean Zernike-moment amplitudes of all modes. It is obvious that the same shape features of the double mode 1 and 2; 4 and 5; etc. are now show 100% correlation according to the Zernike moment descriptors. Besides, the Zernike moments of the double mode have zero correlation with any other mode shapes.
Furthermore, as discussed in Section 2 and 3, the rotational difference between the double modes can be revealed by equation (25). Figure 8 demonstrates an example where (a) shows the shape pattern of mode 2 is the same as mode 1 but rotated through $45^\circ$, (b) plots the amplitudes of their corresponding Zernike moments. It is clear that only two Zernike moments are significant - $Z_{2,2}$ is the largest and $Z_{4,2}$ is the second largest which are shown in Figure 7. For the two $Z_{2,2}$ in (a), their phase difference is $90.1^\circ$ and the repetition $m = 2$. Then the angle $\alpha = 90.1^\circ/2 = 45.1^\circ$ which is almost exactly the difference between these two modes. An angle of $43.5^\circ$ is obtained by comparing the angles of the two Zernike moments $Z_{4,2}$ in (b). The error ($45^\circ - 43.5^\circ$) is due to the numerical error during the discretization of the Zernike polynomials. Similar results are obtained for mode pair 4 and 5 ($\alpha = 30^\circ$), 6 and 7 ($\alpha = 90.1^\circ$), etc.
Figure 6: Comparison Between Original and Reconstructed Mode Shapes (Free-Free Disc)

Figure 7: The $Z_{2,2}$ and $Z_{4,2}$ ZMD of Mode 1 and 2 (Free-Free)

(a) Patterns of Double Mode Shapes 1 and 2

(b) Amplitude of the Zernike Moments

Figure 8: Mode Shape Pattern and ZMD of Mode 1 and 2 (Free-Free)
3.2 Rectangular Plate

Since the Zernike moments are defined over a unit circle, mapping to a unit circle should be carried out before applying the ZMD for other structures, e.g. a rectangular plate, in order to preserve the complete shape features of the original structure. For example, a polygon shown in Figure 9 may be mapped into a circle by the following definitions,

1. The centroid of the polygon is mapped to the centre of the circle;
2. The triangles formed by the centroid and two neighbouring vertices in the polygon are mapped to the sectors in the circle;
3. The angle of each sector is determined by the area percentage of the corresponding triangle in the polygon as shown in Figure 10.

According to the definitions above, the polygon may be mapped onto a circle by the following equations [20],

$$
\begin{align*}
\rho &= \frac{(y_B - y_A)x - (x_B - x_A)y}{x_A y_B - x_B y_A} R \\
\theta &= \frac{x_A y_B - x_B y_A}{(y_B - y_A)x - (x_B - x_A)y} \left( \Gamma_{B'} - \Gamma_A' \right) + \Gamma_A' 
\end{align*}
$$

A 20mm×30mm×2mm rectangular plate was modelled with 600 four-node quadrilateral elements and 651 nodes. The first 12 mode shapes are considered. The coordinate mappings were performed before applying the ZMD as shown in Figure 11. The ZMD based on the circular mode-shape patterns are shown in Figure 12 and 13 up to order 8 which is sufficient for describing the first 12 modes according to the correlation between the original and reconstructed mode shape patterns as show in Figure 14.

Furthermore, the symmetries of the mode shapes may also be indicated by the Zernike moments [20]. e.g. If the image is x-symmetric, the imaginary part of the Zernike moment is zero and vice versa. i.e.

$$
I(\rho, \vartheta) = I(\rho, -\vartheta) \Leftrightarrow \Im(Z_{n,m}) = 0, \forall m
$$

If the image is x-antisymmetric, the real part of the Zernike moment is zero and vice versa. i.e.

$$
I(\rho, \vartheta) = -I(\rho, -\vartheta) \Leftrightarrow \Re(Z_{n,m}) = 0, \forall m
$$

Therefore, the symmetry of the mode-shape pattern in Figure 11 may clearly be indicated by the Zernike moments in the Figure 12 which are aligned exactly on the orthogonal x-y axes corresponding to the real and imaginary components of the Zernike moments.
4. CONCLUSION

Vibration mode-shape recognition based on image processing and pattern recognition has been considered in this paper. In particular, the Zernike moment descriptor was investigated and applied to example problems including the modes of circular and rectangular plates. Results demonstrate that the ZMD is especially powerful in the discrimination of circular and spherical shapes. It may be a good supplement or even replacement of the conventional MAC for such structures because the ZMD not only provides the closeness between the predicted and measured mode shapes but also reveals the subtle difference between them. More general and comprehensive shape recognition techniques are presently being developed and may be used together with finite element model updating especially for large and complicated structures.
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